

Optimal constant in an  $L^2$  extension problem  
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**Abstract** In this paper, we solve the optimal constant problem in the setting of Ohsawa's generalized  $L^2$  extension theorem. As applications, we prove a conjecture of Ohsawa and the extended Suita conjecture, we also establish some relations between Bergman kernel and logarithmic capacity on compact and open Riemann surfaces.

**Keywords**  $L^2$  extension theorem, optimal  $L^2$  estimate, Bergman kernel, a conjecture of Ohsawa, extended Suita conjecture

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## 1 Introduction and main results

### 1.1 Introduction and background

Cartan's theorems A and B on Stein manifolds are important and fundamental in several complex variables. An equivalent version is an extension theorem as follows. Given a holomorphic vector bundle on a Stein manifold and a closed complex subvariety in the Stein manifold, then any section of the restriction of the bundle to the complex subvariety can be holomorphically extended to the section of the bundle over the Stein manifold. A natural question is that if the holomorphic section of the bundle on the complex subvariety is of a special property (say, invariant w.r.t. a group action or bounded or  $L^2$ ), could the holomorphic extension be still of the special property? For the case of invariant version of the extension theorem, the reader is referred to [45]. In the present paper, we deal with the question for the case of  $L^2$  extension.

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### 1.1.1 Optimal constant problem in the $L^2$ extension theorem for hypersurfaces

Ohsawa and Takegoshi [33] obtained the famous  $L^2$  extension theorem without negligible weight for the hypersurface with a holomorphic defining function. In the setting of Ohsawa and Takegoshi, the optimal constant problem in the  $L^2$  extension theorem was widely discussed for various cases by many authors such as Berndtsson, Blocki, Demailly, Guan-Zhou-Zhu, Manivel, McNeal-Varolin, Siu etc., in [2, 5, 6, 11, 20, 25, 27, 28, 33, 36, 38, 47]. After his joint paper with Takegoshi, Ohsawa in [29] obtained the  $L^2$  extension theorem on pseudoconvex domains in  $\mathbb{C}^n$  with negligible weights for hypersurface with a holomorphic defining function. In the setting of Ohsawa in [29], continuing our work [47], we in [18] obtained the optimal constant version of the  $L^2$  extension theorem with negligible weights on Stein manifolds as follows.

**Theorem 1.1** ([18]). *Let  $X$  be a Stein manifold of dimension  $n$ . Let  $\varphi$  and  $\psi$  be plurisubharmonic functions on  $X$ . Assume that  $w$  is a holomorphic function on  $X$  such that  $\sup_X(\psi + 2\log|w|) \leq 0$  and  $dw$  does not vanish identically on any branch of  $w^{-1}(0)$ . Denote  $H = w^{-1}(0)$  and  $H_0 = \{x \in H : dw(x) \neq 0\}$ . Then there exists a uniform constant  $C = 1$  such that, for any holomorphic  $(n-1)$ -form  $f$  on  $H_0$  satisfying*

$$c_{n-1} \int_{H_0} e^{-\varphi-\psi} f \wedge \bar{f} < \infty,$$

where  $c_k = (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^k$  for  $k \in \mathbb{Z}$ , there exists a holomorphic  $n$ -form  $F$  on  $X$  satisfying  $F = dw \wedge \tilde{f}$  on  $H_0$  with  $\iota^* \tilde{f} = f$  and

$$c_n \int_X e^{-\varphi} F \wedge \bar{F} \leq 2C\pi c_{n-1} \int_{H_0} e^{-\varphi-\psi} f \wedge \bar{f},$$

where  $\iota : H_0 \rightarrow X$  is the inclusion map.

Our proof in [18] was based on the methods of [20] and [47]. In the present paper, we will consider some generalization of the above theorem.

### 1.1.2 Suita conjecture

Suita conjecture (see [42]), which was posed originally on open Riemann surfaces in 1972, was motivated to answer a question posed by Sario and Oikawa in [35] about the relation between the Bergman kernel  $\kappa_\Omega$  for holomorphic  $(1, 0)$  forms on an open Riemann surface  $\Omega$  and logarithmic capacity  $c_\beta(z)$  which is locally defined by  $c_\beta(z) = \exp \lim_{\xi \rightarrow z} (G_\Omega(\xi, z) - \log|\xi - z|)$  on  $\Omega$ , which admits a Green function  $G_\Omega$ . The conjecture is stated below.

**Suita conjecture.**  $(c_\beta(z))^2 |dz|^2 \leq \pi \kappa_\Omega(z)$ , for any  $z \in \Omega$ .

The conjecture is true (see [6, 18]).

### 1.1.3 Ohsawa's generalized $L^2$ extension theorem

In [32], Ohsawa considered an even more general setting than before in [28–30, 33] and proved a general extension theorem (main theorem in [32]), which covers earlier main results in [28–30, 33]:

**Theorem 1.2** ([32]). *Let  $(M, S)$  satisfy condition (ab),  $h$  be a smooth metric on holomorphic vector bundle  $E$  on  $M$  with rank  $r$ . Then, for any function  $\Psi$  on  $M$  such that  $\Psi \in \Delta_{h,\delta}(S) \cap C^\infty(M \setminus S)$ , there exists a uniform constant  $C = \max_{1 \leq k \leq n} \frac{2^8 \pi}{k!}$  such that, for any holomorphic section  $f$  of  $K_M \otimes E|_S$  on  $S$  satisfying*

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty,$$

there exists a holomorphic section  $F$  of  $K_M \otimes E$  on  $M$  satisfying  $F = f$  on  $S$  and

$$\int_M |F|_h^2 dV_M \leq C(1 + \delta^{-\frac{3}{2}}) \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

Especially, if  $\Psi \in \Delta(S) \cap \Delta_{h,\delta}(S) \cap C^\infty(M \setminus S)$ , then there exists a holomorphic section  $F$  of  $K_M \otimes E$  on  $M$  satisfying  $F = f$  on  $S$  and

$$\int_M |F|_h^2 dV_M \leq C \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

For the sake of completeness, let us recall and explain the symbols and notation in the above theorem as in [32].

Let  $M$  be an  $n$ -dimensional complex manifold, and  $S$  be a closed complex submanifold of  $M$ . Let  $dV_M$  be a continuous volume form on  $M$ . We consider a class of upper-semi-continuous function  $\Psi$  from  $M$  to the interval  $[-\infty, 0)$  such that

- (1)  $\Psi^{-1}(-\infty) \supset S$ , and  $\Psi^{-1}(-\infty)$  is a closed subset of  $M$ ;
- (2) If  $S$  is  $l$ -dimensional around a point  $x$ , there exists a local coordinate  $(z_1, \dots, z_n)$  on a neighborhood  $U$  of  $x$  such that  $z_{l+1} = \dots = z_n = 0$  on  $S \cap U$  and

$$\sup_{U \setminus S} \left| \Psi(z) - (n-l) \log \sum_{l+1}^n |z_j|^2 \right| < \infty.$$

The set of such polar functions  $\Psi$  will be denoted by  $\#(S)$ .

For each  $\Psi \in \#(S)$ , one can associate a positive measure  $dV_M[\Psi]$  on  $S$  as the minimum element of the partial ordered set of positive measures  $d\mu$  satisfying

$$\int_{S_l} f d\mu \geq \limsup_{t \rightarrow \infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} 1_{\{-1-t < \Psi < -t\}} dV_M$$

for any nonnegative continuous function  $f$  with  $\text{Supp} f \subset\subset M$ . Here  $S_l$  denotes the  $l$ -dimensional component of  $S$ , and  $\sigma_m$  denotes the volume of the unit sphere in  $\mathbb{R}^{m+1}$ .

Let  $\omega$  be a Kähler metric on  $M \setminus (X \cup S)$ . We can also define measure  $dV_\omega[\Psi]$  on  $S \setminus X$  as the minimum element of the partial ordered set of positive measures  $d\mu'$  satisfying

$$\int_{S_l} f d\mu' \geq \limsup_{t \rightarrow \infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_{M \setminus (X \cup S)} f e^{-\Psi} 1_{\{-1-t < \Psi < -t\}} dV_\omega$$

for any nonnegative continuous function  $f$  with  $\text{Supp}(f) \subset\subset M \setminus X$  (as  $\text{Supp}(1_{\{-1-t < \Psi < -t\}}) \cap \text{Supp}(f) \subset\subset M \setminus (X \cup S)$ , the right-hand side of the above inequality is well-defined).

Let  $u$  be a continuous section of  $K_M \otimes E$ , where  $E$  is a holomorphic vector bundle equipped with a continuous metric  $h$  on  $M$ . We define

$$|u|_h^2|_V := \frac{c_n h(e, e) v \wedge \bar{v}}{dV_M},$$

where  $u|_V = v \otimes e$  for an open set  $V \subset M \setminus X$ ,  $v$  is a continuous section of  $K_M|_V$  and  $e$  is a continuous section of  $E|_V$  (especially, we define

$$|u|^2|_V := \frac{c_n u \wedge \bar{u}}{dV_M},$$

when  $u$  is a continuous section of  $K_M$ ). It is clear that  $|u|_h^2$  is independent of the choice of  $V$ . Actually by the following remark, one may see the following relationship between  $dV_\omega[\Psi]$  and  $dV_M[\Psi]$  (resp.  $dV_\omega$  and  $dV_M$ ), precisely,

$$\int_{M \setminus X} f |u|_{h,\omega}^2 dV_\omega[\Psi] = \int_{M \setminus X} f |u|_h^2 dV_M[\Psi], \quad (1.1)$$

$$\left( \text{resp. } \int_{M \setminus X} f |u|_{h,\omega}^2 dV_\omega = \int_{M \setminus X} f |u|_h^2 dV_M \right), \quad (1.2)$$

where  $f$  is a continuous function with compact support on  $M \setminus X$ .

**Remark 1.3.** For the neighborhood  $U$ , let  $u|_U = v \otimes e$ . Note that

$$\begin{aligned} \int_{M \setminus X} f 1_{\{-1-t < \Psi < -t\}} |u|_{h,\omega}^2 e^{-\Psi} dV_\omega &= \int_{M \setminus X} f 1_{\{-1-t < \Psi < -t\}} h(e, e) c_n v \wedge \bar{v} e^{-\Psi} \\ &= \int_{M \setminus X} f 1_{\{-1-t < \Psi < -t\}} |u|_h^2 e^{-\Psi} dV_M, \end{aligned} \quad (1.3)$$

and respectively,

$$\int_{M \setminus (X \cup S)} f |u|_{h,\omega}^2 e^{-\Psi} dV_\omega = \int_{M \setminus (X \cup S)} f h(e, e) c_n v \wedge \bar{v} e^{-\Psi} = \int_{M \setminus (X \cup S)} f |u|_h^2 e^{-\Psi} dV_M, \quad (1.4)$$

where  $f$  is a continuous function with compact support on  $M \setminus X$ . As

$$\text{Supp}(1_{\{-1-t < \Psi < -t\}}) \cap \text{Supp}(f) \subset \subset M \setminus (X \cup S),$$

equality (1.3) is well-defined. Then we have equalities (1.1) and (1.2).

It is clear that  $|u|_h^2$  is independent of the choice of  $V$ , while  $|u|_h^2 dV_M$  is independent of the choice of  $dV_M$  (resp.  $|u|_h^2 dV_M[\Psi]$  is independent of the choice of  $dV_M$ ). Then the space of  $L^2$  integrable holomorphic section of  $K_M$  is denoted by  $A^2(M, K_M, dV_M^{-1}, dV_M)$  (resp. the space of holomorphic section of  $K_M|_S$  which is  $L^2$  integrable with respect to the measure  $dV_M[\Psi]$  is denoted by  $A^2(S, K_M|_S, dV_M^{-1}, dV_M[\Psi])$ ).

**Definition 1.4.** Let  $M$  be an  $n$ -dimensional complex manifold with a continuous volume form  $dV_M$ , and  $S$  be a closed complex submanifold of  $M$ . We call the data  $(M, S)$  satisfy the condition (ab) if  $M$  and  $S$  satisfy the following conditions:

There exists a closed subset  $X \subset M$  such that:

(a)  $X$  is locally negligible with respect to  $L^2$  holomorphic functions, i.e., for any local coordinate neighborhood  $U \subset M$  and for any  $L^2$  holomorphic function  $f$  on  $U \setminus X$ , there exists an  $L^2$  holomorphic function  $\tilde{f}$  on  $U$  such that  $\tilde{f}|_{U \setminus X} = f$  with the same  $L^2$  norm.

(b)  $M \setminus X$  is a Stein manifold which intersects with every component of  $S$ .

**Remark 1.5.** In fact, the condition (ab) is the same as condition (1) in Theorem 4 in [32]. The data  $(M, S)$  with the condition (ab) includes all the following well-known examples:

(1)  $M$  is a Stein manifold (including open Riemann surfaces), and  $S$  is any closed complex submanifold of  $M$ ;

(2)  $M$  is a complex projective algebraic manifold (including compact Riemann surfaces), and  $S$  is any closed complex submanifold of  $M$ ;

(3)  $M$  is a projective family (see [38]), and  $S$  is any closed complex submanifold of  $M$ .

The following remark shows the extension properties of holomorphic sections of holomorphic vector bundles from  $M \setminus X$  to  $M$ .

**Remark 1.6.** Let  $(M, S)$  satisfy the condition (ab),  $h$  be a singular metric on holomorphic line bundle  $L$  on  $M$  (resp. continuous metric on holomorphic vector bundle  $E$  on  $M$  with rank  $r$ ), where  $h$  has locally positive lower bound. Let  $F$  be a holomorphic section of  $K_{M \setminus X} \otimes E|_{M \setminus X}$ , which satisfies  $\int_{M \setminus X} |F|_h^2 < \infty$ .

As  $h$  has locally positive lower bound and  $M$  satisfies (a) of the condition (ab), there is a holomorphic section  $\tilde{F}$  of  $K_M \otimes L$  on  $M$  (resp.  $K_M \otimes E$ ), such that  $\tilde{F}|_{M \setminus X} = F$ .

Let  $\Delta_{h,\delta}(S)$  be the subset of function  $\Psi$  in  $\#(S)$  which satisfies  $\Theta_{he^{-\Psi}} \geq 0$  and  $\Theta_{he^{-(1+\delta)\Psi}} \geq 0$  on  $M \setminus S$  in the sense of Nakano.

Let  $\Delta(S)$  be the subset of plurisubharmonic functions  $\Psi$  in  $\#(S)$ ,  $\varphi$  be a locally integrable function on  $M$ . Let  $\Delta_{\varphi,\delta}(S)$  be the subset of functions  $\Psi$  in  $\#(S)$ , such that  $\Psi + \varphi$  and  $(1 + \delta)\Psi + \varphi$  are both plurisubharmonic functions on  $M$ .

## 1.2 Main results

In the present paper, we give some generalizations of both Theorems 1.1 and 1.2.

### 1.2.1 Optimal constant in the generalized $L^2$ extension theorem for non-smooth polar function

In the following theorem, we give an optimal constant of a generalization of Theorem 1.2 for trivial line bundle and non-plurisubharmonic polar function  $\Psi$ . After that, we show that the constant  $C$ , which is equal to 1, is optimal.

**Theorem 1.7** (Main theorem 1). *Let  $(M, S)$  satisfy condition (ab),  $\varphi$  be a continuous function on  $M$ . Then, for negative function  $\Psi$  on  $M$  satisfying  $\Psi \in \Delta_{\varphi, \delta}(S)$ , there exists a uniform constant  $C = 1$ , such that, for any holomorphic section  $f$  of  $K_M|_S$  on  $S$  satisfying*

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi] < \infty,$$

*there exists a holomorphic section  $F$  of  $K_M$  on  $M$  satisfying  $F = f$  on  $S$  and*

$$\int_M |F|^2 e^{-\varphi} dV_M \leq C(1 + \delta^{-1}) \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi],$$

*where  $|f|^2 := \frac{c_n f \wedge \bar{f}}{dV_M}$ .*

**Remark 1.8.** We will see that the above constant is optimal in the proof. Especially, we will illustrate that for any given planar domain  $\Omega$  in  $\mathbb{C}$  and point  $z_0 \in \Omega$ , the above constant is also optimal, where  $S = \{z_0\}$ .

Note that for any holomorphic line bundle  $L$  on Stein manifold  $M \setminus X$ , we can choose a complex hypersurface  $H$  on  $M \setminus X$ , such that  $L|_{M \setminus (X \cup H)}$  is a trivial line bundle, where  $M \setminus (X \cup H)$  is a Stein manifold, and  $H$  does not contain any component of  $S \setminus X$ . By Remark 1.6, we obtain the following corollary:

**Corollary 1.9.** *Let  $(M, S)$  satisfy condition (ab),  $L$  be a holomorphic line bundle on  $M$  with a continuous metric  $h$  (resp. a singular metric  $h$  satisfying  $\Theta_h \geq \omega$ , where  $\omega$  is a smooth real  $(1,1)$ -form on  $M$ ). Then, for negative function  $\Psi \in \#(S)$  on  $M$  satisfying  $\Theta_{he^{-\Psi}} \geq 0$  and  $\Theta_{h e^{-(1+\delta)\Psi}} \geq 0$  (resp.  $\sqrt{-1}\partial\bar{\partial}\Psi + \omega > 0$  and  $(1+\delta)\sqrt{-1}\partial\bar{\partial}\Psi + \omega > 0$ ) in the sense of current on  $M$ , there exists a uniform constant  $C = 1$ , such that, for any holomorphic section  $f$  of  $K_M \otimes L|_S$  on  $S$  satisfying*

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty,$$

*there exists a holomorphic section  $F$  of  $K_M \otimes L$  on  $M$  satisfying  $F = f$  on  $S$  and*

$$\int_M |F|_h^2 dV_M \leq C(1 + \delta^{-1}) \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

### 1.2.2 Some relations between Bergman kernel and logarithmic capacity on compact Riemann surfaces

Let  $X$  be a compact Riemann surface with genus  $g \geq 2$  (resp.  $g = 1$  complex torus, which is denoted by  $X_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  ( $\tau \in \mathbb{C}$ ,  $\Im\tau > 0$ )). Then there exists a conformal metric  $\omega$  on  $X$ , obtained by descending the Poincaré metric from its universal covering space to itself, such that  $\text{curv}\omega = -1$  (resp. the Kähler metric  $\omega := \frac{1}{\Im\tau} dz \otimes d\bar{z}$ ).

Consider the function  $g(p, q) : X \times X \rightarrow [-\infty, 0)$ , such that for each fixed  $q \in X$ :

- (a)  $\Delta_\omega g(\cdot, q) = -1$  on  $X \setminus q$  (here  $\Delta_\omega$  is the Laplacian with respect to the metric  $\omega$ );
- (b)  $g(p, q) = \log \text{dist}_\omega(p, q) + O(1)$ , as  $p \rightarrow q$ ;
- (c)  $g(p, p) = -\infty$ .

The existence and uniqueness of such a function for general compact Riemann surfaces are proved by Arakelov [1]. And define further  $c_X(p) := \exp(\lim_{q \rightarrow p} (g(p, q) - \log \text{dist}_\omega(p, q)))$  to be the logarithmic capacity with respect to  $p$ .

By Corollary 1.9, we can obtain the relations between Bergman kernel and logarithmic capacity on Riemann surface with genus  $g \geq 2$  as follows:

**Theorem 1.10.** Let  $\kappa_{X,m}$  be the Bergman kernel of holomorphic line bundle  $mK_X$  ( $m \geq 2$ ), where  $X$  is a compact Riemann surface with genus  $g \geq 2$ . Then we have

$$\pi \left( 1 + \frac{1}{(g-1)(m-1)-1} \right) |\kappa_{X,m}(p, p)|_{h^m} \geq c_X^2(p), \quad (1.5)$$

where  $h$  is the Hermitian metric on  $K_X$  induced by Poincaré metric on  $X$  (see the proof of the present theorem).

As Arakelov metric  $ds_A^2 := (\exp(2 \lim_{\xi \rightarrow z} (g(\xi, z) - \log |\xi - z|))) |dz|^2$  (see [43]), it is clear that  $c_X^2 = \frac{ds_A^2}{\omega}$ . Note that  $\kappa_{X,m}$  is the maximum value of holomorphic sections on  $mK_X$  whose  $L^2$  norm is 1 with respect to  $\omega$ . Then the above theorem reveals a relation (only depends on  $n$  and  $g$ ) between two important objects appearing in string perturbation theory: Arakelov metric and holomorphic sections on  $mK_X$ , where Arakelov metric plays an important role in bosonization (see [22]), and holomorphic section on  $mK_X$  corresponds to the zero-mode in string perturbation theory (see [22]).

By Corollary 1.9, we also can obtain a relation between Bergman kernel and logarithmic capacity on compact Riemann surface with genus  $g = 1$  as follows:

**Theorem 1.11.** Let  $\kappa_{X,d}$  be the Bergman kernel of line bundle  $K_X \otimes L$ , where  $X$  is a compact Riemann surface with genus  $g = 1$ , and  $L$  is positive line bundle with degree  $d$  ( $d > 2$ ). Then we have

$$\pi \left( 1 + \frac{1}{\frac{d}{2} - 1} \right) |\kappa_{X,d}(p, p)|_{\omega, h_L} \geq c_X^2(p), \quad (1.6)$$

where  $h_L$  is the canonical Hermitian metric on  $L$  which satisfies  $c_1(L) = b\omega$  ( $b > 0$ ).

### 1.2.3 Optimal constant in the generalized $L^2$ extension theorem on line bundles with singular metric and non-smooth polar function

In the following theorem, we give a generalization of Theorem 1.2 on canonical line bundles and plurisubharmonic polar function  $\Psi$  and show that the optimal constant  $C$  is equal to 1.

**Theorem 1.12.** Let  $(M, S)$  satisfy condition (ab),  $\varphi$  be a plurisubharmonic function on  $M$ . Then, for any negative plurisubharmonic function  $\Psi$  on  $M$  such that  $\Psi \in \Delta(S)$ , there exists a uniform constant  $C = 1$ , such that, for any holomorphic section  $f$  of  $K_M|_S$  on  $S$  satisfying

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi] < \infty,$$

there exists a holomorphic section  $F$  of  $K_M$  on  $M$  satisfying  $F = f$  on  $S$  and

$$\int_M |F|^2 e^{-\varphi} dV_M \leq C \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi].$$

By retraction (see Lemma 2.6) and convolution, and Remark 1.6, we can obtain the following corollary:

**Corollary 1.13.** Let  $(M, S)$  satisfy condition (ab),  $L$  be a holomorphic line bundle on  $M$  with a singular metric  $h$  satisfying  $\Theta_h \geq 0$  in the sense of current. Then, for negative plurisubharmonic function  $\Psi$  on  $M$  satisfying  $\Psi \in \Delta(S)$ , there exists a uniform constant  $C = 1$ , such that, for any holomorphic section  $f$  of  $K_M \otimes L|_S$  on  $S$  satisfying

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty,$$

there exists a holomorphic section  $F$  of  $K_M \otimes L$  on  $M$  satisfying  $F = f$  on  $S$  and

$$\int_M |F|_h^2 dV_M \leq C \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

### 1.2.4 A conjecture of Ohsawa

If  $\Delta(S)$  is non-empty, we set  $G(z, S) := (\sup\{u(z) : u \in \Delta(S)\})^*$ , which is the upper envelope of  $\sup\{u(z) : u \in \Delta(S)\}$ . It is clear that  $G(z, S)$  is a plurisubharmonic function on  $M$  (see Choquet's lemma (Lemma 4.23 in [12])). By Proposition 9 in [32], we have  $G(z, S) \in \Delta(S)$ .  $G(z, S)$  is called generalized pluricomplex Green function on  $M$  with poles on  $S$ . If  $\Delta(S)$  is empty,  $G(z, S) := -\infty$ . When  $S = \{z\}$  for some  $z \in M$ ,  $G(z, S)$  is the so-called the pluricomplex Green function (see [10]).

Let  $(M, S)$  satisfy condition (ab),  $G(\cdot, S)$  be the generalized pluricomplex Green function which is nontrivial. Let  $dV_M$  be a continuous volume form on  $M$  and let  $\{\sigma_j\}_{j=1}^\infty$  (resp.  $\{\tau_j\}_{j=1}^\infty$ ) be a complete orthogonal system of  $A^2(M, K_M, dV_M^{-1}, dV_M)$  (resp.  $A^2(S, K_M|_S, dV_M^{-1}, dV_M[G(\cdot, S)])$ ) and put  $\kappa_M = \sum_{j=1}^\infty \sigma_j \otimes \bar{\sigma}_j \in C^\omega(M, K_M \otimes \bar{K}_M)$  (resp.  $\kappa_{M/S} = \sum_{j=1}^\infty \tau_j \otimes \bar{\tau}_j \in C^\omega(S, K_M \otimes \bar{K}_M)$ ).

One motivation to estimate the constant  $C$  in Theorem 1.7 comes from the conjecture of Ohsawa (see [32]) on  $(M, S)$  satisfying condition (ab) and admitting nontrivial generalized pluricomplex Green functions on  $M$  with poles on  $S$ , which is stated below:

**Conjecture (Ohsawa).**  $(\pi^k/k!)\kappa_M(x) \geq \kappa_{M/S}(x)$  for any  $x \in S_{n-k}$ .

The relationship between the conjecture of Ohsawa and the extension theorem was observed and explored by Ohsawa [32], and he proved the estimate with  $C = \frac{2^s \pi}{\pi^k/k!}$ . The conjecture of Ohsawa can be seen as an extension of Suita conjecture (see Section 3 of [32]).

Using Theorem 1.12, we get

**Corollary 1.14.** *The above Ohsawa's conjecture holds.*

### 1.2.5 Extended Suita conjecture

Given a weight  $\rho$ , one may define a weighted Bergman space consisting  $w$ , such that  $\int_\Omega \rho w \wedge \bar{w} < +\infty$ . Denote its Bergman kernel by  $\kappa_{\Omega, \rho}$ .

Let  $\Omega$  be an open Riemann surface, which admits a Green function. Let  $h$  be a harmonic function on  $\Omega$ , and  $\rho = e^{-2h(z)}$ . Given a harmonic function, there is an extended Suita conjecture in [44]:

**Extended Suita conjecture.**  $c_\beta^2(p)|dz|^2 \leq \pi \rho(p) \kappa_{\Omega, \rho}(p)$  for any  $p \in \Omega$ .

Let  $z_0 \in \Omega$ , with local coordinate  $z$ . Let  $p : \Delta \rightarrow \Omega$  be the universal covering from unit disc  $\Delta$  to  $\Omega$ . The fundamental group of  $\Omega$  naturally acts on Prym differential, whose set is denoted by  $\Gamma^\chi(\Omega)$ . Actually this is equivalent to the following equivariant version in term of multiplicative Bergman kernel denoted by  $\kappa_\Omega^\chi$ , where  $\chi$  is a representation of the fundamental group (see [44]).

**Extended Suita conjecture.**  $c_\beta^2(z_0) \leq \pi B_\Omega^\chi(z_0)$ .

Using Theorem 1.12, we get

**Corollary 1.15.** *The above extended Suita conjecture holds.*

**Remark 1.16.** When  $h \equiv 0$ , the above reduces to Suita conjecture.

There is an equivalent version of Suita conjecture in terms of Fuchsian groups as follows:

**Remark 1.17.** There is a Fuchsian group  $\Gamma$  without elliptic elements such that  $\Delta/\Gamma$  is conformally equivalent to  $\Omega$ . We can choose  $\Gamma$  such that  $0 \in \Delta$  corresponds to  $\omega \in \Omega$ . When  $\Omega$  admits a Green function, the group  $\Gamma$  is of convergence type. In [34], Pommerenke and Suita proved that Suita conjecture is equivalent to

$$\sum_{\gamma \in \Gamma} \gamma'(0) \geq \prod_{\gamma \in \Gamma, \gamma \neq \iota} |\gamma(0)|^2, \quad (1.7)$$

where  $\iota$  denotes the identity.

### 1.2.6 Boundary behavior of the quotient of logarithmic capacity and Bergman kernel

In this subsection, we discuss boundary behavior of the quotient of logarithmic capacity and Bergman kernel by squeezing function  $s_\Omega$  on bounded planar domain  $\Omega$  as follows:



**Definition 1.18** ([15]). Let  $\Omega$  be a bounded domain in  $\mathbb{C}$ . For  $p \in \Omega$  and an (open) holomorphic embedding  $f : \Omega \rightarrow \Delta$  with  $f(p) = 0$ , we define

$$s_{\Omega}(p, f) = \sup\{r \mid \Delta(0, r) \subset f(\Omega)\},$$

and the squeezing number  $s_{\Omega}(p)$  of  $D$  at  $p$  is defined as

$$s_{\Omega}(p) = \sup_f \{s_{\Omega}(p, f)\},$$

where the supremum is taken over all holomorphic embeddings  $f : \Omega \rightarrow \Delta$  with  $f(p) = 0$ ,  $\Delta$  is the unit ball in  $\mathbb{C}$  and  $\Delta(0, r)$  is the ball in  $\mathbb{C}$  with center 0 and radius  $r$ . We call  $s_{\Omega}$  the *squeezing function* on  $\Omega$ .

Let  $C(\Omega, z) := \frac{c_{\beta}(z)^2 |dz|^2}{\pi \kappa_{\Omega}(z)}$ , where  $z \in \Omega$ . We consider the boundary behavior of  $C(\Omega, z)$  as follows:

**Proposition 1.19.** Let  $\Omega$  be a Riemann surface which is biholomorphic equivalent to a bounded planar domain. Then we have  $1 \geq C(\Omega, z) \geq s_{\Omega}^2$ . If  $\lim_{z \rightarrow \partial\Omega} s_{\Omega} = 1$ , then  $\lim_{z \rightarrow \partial\Omega} C(\Omega, z) = 1$ .

Moreover, when  $\Omega$  has smooth boundary, by Theorem 5.2 in [15], we have  $\lim_{z \rightarrow \partial\Omega} s_{\Omega} = 1$ . By Remark 1.16, we obtain  $C(\Omega, z) \leq 1$ . Then we have

**Corollary 1.20** ([42]). Let  $\Omega$  be a Riemann surface which is biholomorphically equivalent to a bounded planar domain with smooth boundary. Then we have

$$\lim_{z \rightarrow \partial\Omega} C(\Omega, z) = 1.$$

### 1.2.7 Optimal constant in the generalized $L^2$ extension theorem on holomorphic vector bundle and polar function which is smooth outside $S$

In the following theorem, we give a generalization of Theorem 1.2 on holomorphic vector bundles and polar function  $\Psi$  which is smooth outside  $S$ , and show that the optimal constant  $C$  is equal to 1.

**Theorem 1.21** (Main theorem 2). Let  $(M, S)$  satisfy condition (ab),  $h$  be a smooth metric on a holomorphic vector bundle  $E$  on  $M$  with rank  $r$ . Then, for any function  $\Psi$  on  $M$  such that  $\Psi \in \Delta_{h, \delta}(S) \cap C^{\infty}(M \setminus S)$ , there exists a uniform constant  $C = 1$  such that, for any holomorphic section  $f$  of  $K_M \otimes E|_S$  on  $S$  satisfying

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty,$$

there exists a holomorphic section  $F$  of  $K_M \otimes E$  on  $M$  satisfying  $F = f$  on  $S$  and

$$\int_M |F|_h^2 dV_M \leq C(1 + \delta^{-1}) \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$

When  $\delta$  approaches to infinity, we have a generalization of Theorem 4 in [32] (the main theorem) of holomorphic vector bundles in plurisubharmonic case ( $\Psi$  is plurisubharmonic on  $M$ ) and show that the  $C$  is equal to 1, which is optimal.

**Corollary 1.22.** Let  $(M, S)$  satisfy condition (ab),  $h$  be a smooth metric on holomorphic vector bundle  $E$  on  $M$  with rank  $r$ . Then, for any function  $\Psi$  on  $M$  such that  $\Psi \in \Delta(S) \cap \Delta_{h, \delta}(S) \cap C^{\infty}(M \setminus S)$ , there exists a uniform constant  $C = 1$  such that, for any holomorphic section  $f$  of  $K_M \otimes E|_S$  on  $S$  satisfying

$$\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty,$$

there exists a holomorphic section  $F$  of  $K_M \otimes E$  on  $M$  satisfying  $F = f$  on  $S$  and

$$\int_M |F|_h^2 dV_M \leq C \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi].$$



## 2 Some lemmas used in the proof of main theorems

In this section, we give some lemmas which will be used in the proofs of main theorems of the present paper.

### 2.1 $L^2$ estimates for $\bar{\partial}$ equations

In this subsection, we give some lemmas on  $L^2$  estimates for some  $\bar{\partial}$  equations, and  $\bar{\partial}^*$  means the Hilbert adjoint operator of  $\bar{\partial}$ .

**Lemma 2.1.** *Let  $(X, ds_X^2)$  be a Kähler manifold of dimension  $n$  with a Kähler metric  $ds_X^2$ ,  $\Omega \subset\subset X$  be a domain with  $C^\infty$  boundary  $b\Omega$ ,  $\Phi \in C^\infty(\bar{\Omega})$ . Let  $\rho$  be a  $C^\infty$  defining function for  $\Omega$  such that  $|d\rho| = 1$  on  $b\Omega$ . Let  $\eta$  be a smooth function on  $\bar{\Omega}$ . Then for any  $(n, 1)$ -form  $\alpha = \sum'_{|I|=n} \alpha_{I\bar{j}} dz^I \wedge d\bar{z}^j \in \text{Dom}_\Omega(\bar{\partial}^*) \cap C^\infty_{(n,1)}(\bar{\Omega})$ ,*

$$\begin{aligned} & \int_\Omega \eta |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} dV_X + \int_\Omega \eta |\bar{\partial} \alpha|^2 e^{-\Phi} dV_X \\ &= \sum'_{|J|=1} \sum_{i,j=1}^n \int_\Omega \eta g^{i\bar{j}} \bar{\nabla}_j \alpha_{I\bar{J}} \overline{\bar{\nabla}_i \alpha^{I\bar{J}}} e^{-\Phi} dV_X + \sum_{i,j=1}^n \int_{b\Omega} \eta (\partial_i \bar{\partial}_j \rho) \alpha_I^i \overline{\alpha^{I\bar{j}}} e^{-\Phi} dS \\ &+ \sum_{i,j=1}^n \int_\Omega \eta (\partial_i \bar{\partial}_j \Phi) \alpha_I^i \overline{\alpha^{I\bar{j}}} e^{-\Phi} dV_X + \sum_{i,j=1}^n \int_\Omega -(\partial_i \bar{\partial}_j \eta) \alpha_I^i \overline{\alpha^{I\bar{j}}} e^{-\Phi} dV_X \\ &+ 2\text{Re}(\bar{\partial}_\Phi^* \alpha, \alpha_\perp (\bar{\partial} \eta)^\sharp)_{\Omega, \Phi}, \end{aligned} \quad (2.1)$$

where  $(g^{i\bar{j}})_{n \times n} = \overline{(g_{i\bar{j}})}_{n \times n}^{-1}$ , and  $dV_X$  is the volume form with  $ds_X^2$ .

The symbols and notation are referred to [47]. See also [7, 36, 38] and [41].

**Lemma 2.2.** *Let  $(X, ds_X^2)$  be a Hermitian manifold of dimension  $n$  with a Hermitian metric  $ds_X^2$ ,  $\Omega \subset\subset X$  be a strictly pseudoconvex domain in  $X$  with  $C^\infty$  boundary  $b\Omega$  and  $\Phi \in C^\infty(\bar{\Omega})$ . Let  $\lambda$  be a  $\bar{\partial}$  closed smooth form of bi-degree  $(n, 1)$  on  $\bar{\Omega}$ . Assume the inequality*

$$|(\lambda, \alpha)_{\Omega, \Phi}|^2 \leq C \int_\Omega |\bar{\partial}_\Phi^* \alpha|^2 \frac{e^{-\Phi}}{\mu} dV_X < \infty,$$

holds for all  $(n, 1)$ -form  $\alpha \in \text{Dom}_\Omega(\bar{\partial}^*) \cap \text{Ker}(\bar{\partial}) \cap C^\infty_{(n,1)}(\bar{\Omega})$ , where  $\frac{1}{\mu}$  is an integrable positive function on  $\Omega$  and  $C$  is a constant. Then there is a solution  $u$  to the equation  $\bar{\partial} u = \lambda$  such that

$$\int_\Omega |u|^2 \mu e^{-\Phi} dV_X \leq C.$$

The proof is similar to the proof of Lemma 2.4 in [2].

**Lemma 2.3** (See [14, 29]). *Let  $(X, \omega)$  be a Kähler manifold of dimension  $n$  with a Kähler metric  $\omega$ ,  $E$  be a holomorphic hermitian vector bundle over  $X$ , and  $\Omega \subset\subset X$  be a domain with  $C^\infty$  boundary  $b\Omega$ . Let  $\eta, g > 0$  be smooth functions on  $X$ . Then for every form  $\alpha \in \mathcal{D}(X, \Lambda^{n,q} T_X^* \otimes E)$ , which is the space of smooth differential  $(n, q)$ -forms with values in  $E$  with compact support, we have*

$$\|(\eta + g^{-1})^{\frac{1}{2}} D''^* \alpha\|^2 + \|\eta^{\frac{1}{2}} D'' \alpha\|^2 \geq \langle [\eta \sqrt{-1} \Theta_E - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega] \alpha, \alpha \rangle. \quad (2.2)$$

**Lemma 2.4** (See [11, 14]). *Let  $X$  be a complete Kähler manifold equipped with a (not necessarily complete) Kähler metric  $\omega$ , and let  $E$  be a holomorphic hermitian vector bundle over  $X$ . Assume that there are smooth and bounded functions  $\eta, g > 0$  on  $X$  such that the (Hermitian) curvature operator*

$$B := [\eta \sqrt{-1} \Theta_E - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega]$$

is positive definite everywhere on  $\Lambda^{n,q}T_X^* \otimes E$ , for some  $q \geq 1$ . Then for every form  $\lambda \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$  such that  $D''\lambda = 0$  and  $\int_X \langle B^{-1}\lambda, \lambda \rangle dV_\omega < \infty$ , there exists  $u \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$  such that  $D''u = \lambda$  and

$$\int_X (\eta + g^{-1})^{-1} |u|^2 dV_\omega \leq \int_X \langle B^{-1}\lambda, \lambda \rangle dV_\omega.$$

## 2.2 Properties of Stein manifolds

In this subsection, we recall some well-known properties on Stein manifolds.

**Lemma 2.5** ([17]). *Let  $X$  be a Stein manifold and  $\varphi \in PSH(X)$ . Then there exists a sequence  $\{\varphi_n\}_{n=1,2,\dots}$  of smooth strongly plurisubharmonic functions such that  $\varphi_n \downarrow \varphi$ .*

**Lemma 2.6.** *Let  $X$  be a Stein manifold and  $H$  be a closed complex submanifold of  $X$ . Then there is an open neighborhood  $U$  of  $H$  in  $X$  and a holomorphic retraction  $r : U \rightarrow H$ .*

## 2.3 Properties of polar functions

In this subsection, we collect some lemmas on properties of polar functions.

**Lemma 2.7.** *Let  $M$  be a complex manifold of dimension  $n$ , and  $S$  be an  $(n-1)$ -dimensional submanifold. Let  $\Psi \in \Delta(S)$ . There exists a local coordinate  $(z_1, \dots, z_n)$  on a neighborhood  $U$  of  $x$  such that  $z_n = 0$  on  $S \cap U$  and such that  $\psi := \Psi - \log |z_n|^2$  is continuous on  $U$ . Then we have  $d\lambda_z[\Psi] = e^{-\psi} d\lambda_{z'}$ , where  $d\lambda_z$  and  $d\lambda_{z'}$  denote the Lebesgue measures on  $U$  and  $S \cap U$ .*

*Proof.* Note that  $d\lambda_z[\log |z_n|^2] = d\lambda_{z'}$  for  $z = (z', z_n)$ . By the definition of generalized residue volume form  $d\lambda_z[\Psi]$  and the continuity of  $\psi$ , we can prove the lemma.  $\square$

**Lemma 2.8.** *Let  $M$  be a complex manifold of dimension  $n$ . If  $\Delta(S)$  is non-empty, then  $G(z, S) \in \Delta(S)$ .*

*Proof.* The proof is similar to the proof of Proposition 9 in [32].  $\square$

## 3 Proofs of main theorems

In this section, we give proofs of our main results stated above.

### 3.1 Proof of Theorem 1.7

By Remark 1.6, it suffices to prove the theorem for the case when  $M$  is a Stein manifold and  $S$  is a closed complex submanifold.

Since  $M$  is Stein, one can find a sequence of strictly pseudoconvex domains  $\{D_v\}_{v=1}^\infty$  with smooth boundaries satisfying  $D_v \subset \subset D_{v+1}$  for all  $v$  and  $\bigcup_{v=1}^\infty D_v = M$ . For  $\Psi < 0$ , by Lemma 2.6 and convolutions, we can choose a sequence of smooth functions  $\{\varphi_{v,m}\}_{m=1,2,\dots}$  and  $\{\Psi_{v,m}\}_{m=1,2,\dots}$  on  $D_{v+1}$ , which satisfy  $\varphi_{v,m} + \Psi_{v,m}$  (resp.  $\varphi_{v,m} + (1+\delta)\Psi_{v,m}$ ) decreasing convergent to  $\varphi + \Psi$  (resp.  $\varphi + (1+\delta)\Psi$ ) on  $D_v$  and  $\Psi_{v,m}|_{\overline{D_v}} < 0$ , such that  $\Psi_{v,m} + \varphi_{v,m}$  and  $(1+\delta)\Psi_{v,m} + \varphi_{v,m}$  are both plurisubharmonic functions. Denote  $\Psi_v := \Psi|_{D_v}$ .

Since  $M$  is Stein, there is a holomorphic section  $\tilde{F}$  of  $K_M$  on  $M$  such that  $\tilde{F}|_S = f$ . Let  $ds_M^2$  be a Kähler metric on  $M$ ,  $dV_M$  is the volume form with respect to  $ds_M^2$ .

Let  $\{v_{t_0,\varepsilon}\}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{4})}$  be a family of smooth increasing convex functions on  $\mathbb{R}$ , which are continuous functions on  $\mathbb{R} \cup +\infty$ , such that:

- (1)  $v_{t_0,\varepsilon}(t) = t$  for  $t \geq -t_0 - \varepsilon$ ,  $v_{t_0,\varepsilon}(t) = \text{constant}$  for  $t < -t_0 - 1 + \varepsilon$ ;
- (2)  $v''_{t_0,\varepsilon}(t)$  are pointwise convergent to  $1_{\{-t_0-1 < t < -t_0\}}$ , when  $\varepsilon \rightarrow 0$ , and  $0 \leq v''_{t_0,\varepsilon}(t) \leq 2$  for any  $t \in \mathbb{R}$ ;
- (3)  $v'_{t_0,\varepsilon}(t)$  are pointwise convergent to  $b_{t_0}(t) = \int_{-\infty}^t 1_{\{-t_0-1 < s < -t_0\}} ds$  ( $b_{t_0}$  is also a continuous function on  $\mathbb{R} \cup +\infty$ ), when  $\varepsilon \rightarrow 0$ , and  $0 \leq v'_{t_0,\varepsilon}(t) \leq 1$  for any  $t \in \mathbb{R}$ .

We can construct the family  $\{v_{t_0, \varepsilon}\}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{4})}$  by the setting

$$\begin{aligned} v_{t_0, \varepsilon}(t) := & \int_{-\infty}^t \int_{-\infty}^{t_1} \frac{1}{1-2\varepsilon} 1_{\{-t_0-1+\varepsilon < s < -t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon} ds dt_1 \\ & - \int_{-\infty}^0 \int_{-\infty}^{t_1} \frac{1}{1-2\varepsilon} 1_{\{-t_0-1+\varepsilon < s < -t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon} ds dt_1, \end{aligned} \quad (3.1)$$

where  $\rho_{\frac{1}{4}\varepsilon}$  is the kernel of convolution satisfying  $\text{Supp}(\rho_{\frac{1}{4}\varepsilon}) \subset (-\frac{1}{4}\varepsilon, \frac{1}{4}\varepsilon)$ .

Therefore we have

$$v''_{t_0, \varepsilon}(t) = \frac{1}{1-2\varepsilon} 1_{\{-t_0-1+\varepsilon < t < -t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon},$$

and

$$v'_{t_0, \varepsilon}(t) = \int_{-\infty}^t \frac{1}{1-2\varepsilon} 1_{\{-t_0-1+\varepsilon < s < -t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon} ds.$$

Let  $\eta = s(-v_{t_0, m} \circ \Psi_{v, m})$  and  $\phi = u(-v_{t_0, \varepsilon} \circ \Psi_{v, m})$ , where  $s \in C^\infty((0, +\infty))$  satisfies  $s \geq \frac{1}{\delta}$ , and  $u \in C^\infty((0, +\infty))$  satisfies  $\lim_{t \rightarrow +\infty} u(t) = -\log(1 + \frac{1}{\delta})$ , such that  $u''s - s'' > 0$ , and  $s' - u's = 1$ . Let  $\Phi = \varphi_{v, m} + \Psi_{v, m} + \phi$ .

Now let  $\alpha = \sum_{|I|=n} \sum_{j=1}^n \alpha_{I\bar{j}} dz^I \wedge d\bar{z}^j \in \text{Dom}_{D_v}(\bar{\partial}^*) \cap \text{Ker}(\bar{\partial}) \cap C_{(n,1)}^\infty(\overline{D_v})$ . By Lemma 2.1 and Cauchy-Schwarz inequality, for  $s \geq \frac{1}{\delta}$  and  $\Psi_{v, m} + \varphi_{v, m}$  being a plurisubharmonic function on  $D_{v+1}$ , we get

$$\begin{aligned} \int_{D_v} (\eta + g^{-1}) |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} dV_M & \geq \sum'_{|I|=n} \sum_{j,k=1}^n \int_{D_v} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \Phi - g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_I^j \overline{\alpha^{I\bar{k}}} e^{-\Phi} dV_M \\ & \geq \sum'_{|I|=n} \sum_{j,k=1}^n \int_{D_v} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi + \frac{1}{\delta} \partial_j \bar{\partial}_k (\Psi_{v, m} + \varphi_{v, m}) \\ & \quad - g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_I^j \overline{\alpha^{I\bar{k}}} e^{-\Phi} dV_M, \end{aligned} \quad (3.2)$$

where  $g$  is a positive continuous function on  $D_v$ . We need some calculations in order to determine  $g$ .

We have

$$\begin{aligned} \partial_j \bar{\partial}_k \eta &= -s'(-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \partial_j \bar{\partial}_k (v_{t_0, \varepsilon} \circ \Psi_{v, m}) \\ & \quad + s''(-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \partial_j (v_{t_0, \varepsilon} \circ \Psi_{v, m}) \bar{\partial}_k (v_{t_0, \varepsilon} \circ \Psi_{v, m}), \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \partial_j \bar{\partial}_k \phi &= -u'(-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \partial_j \bar{\partial}_k (v_{t_0, \varepsilon} \circ \Psi_{v, m}) \\ & \quad + u''(-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \partial_j (v_{t_0, \varepsilon} \circ \Psi_{v, m}) \bar{\partial}_k (v_{t_0, \varepsilon} \circ \Psi_{v, m}) \end{aligned} \quad (3.4)$$

for any  $j, k$  which satisfies  $1 \leq j, k \leq n$ .

We have

$$\begin{aligned} & \sum_{1 \leq j, k \leq n} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi - g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_I^j \overline{\alpha^{I\bar{k}}} \\ &= (s' - su') \sum_{1 \leq j, k \leq n} \partial_j \bar{\partial}_k (v_{t_0, \varepsilon} \circ \Psi_{v, m}) \alpha_I^j \overline{\alpha^{I\bar{k}}} \\ & \quad + ((u''s - s'') - gs'^2) \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \bar{\partial}_k (-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \alpha_I^j \overline{\alpha^{I\bar{k}}} \\ &= (s' - su') \sum_{1 \leq j, k \leq n} ((v'_{t_0, \varepsilon} \circ \Psi_{v, m}) \partial_j \bar{\partial}_k \Psi_{v, m} + (v''_{t_0, \varepsilon} \circ \Psi_{v, m}) \partial_j (\Psi_{v, m}) \bar{\partial}_k (\Psi_{v, m})) \alpha_I^j \overline{\alpha^{I\bar{k}}} \\ & \quad + ((u''s - s'') - gs'^2) \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \bar{\partial}_k (-v_{t_0, \varepsilon} \circ \Psi_{v, m}) \alpha_I^j \overline{\alpha^{I\bar{k}}}. \end{aligned} \quad (3.5)$$

For simplicity, we omit composite item  $(-v_{t_0,\varepsilon} \circ \Psi_{v,m})$  after  $s' - su'$  and  $(u''s - s'') - gs'^2$  in the above equalities.

Denote  $g = \frac{u''s - s''}{s'^2} \circ (-v_{t_0,\varepsilon} \circ \Psi_{v,m})$ . We have  $\eta + g^{-1} = (s + \frac{s'^2}{u''s - s''}) \circ (-v_{t_0,\varepsilon} \circ \Psi_{v,m})$ .

Since  $\varphi_{v,m} + \Psi_{v,m}$  and  $\varphi_{v,m} + (1 + \delta)\Psi_{v,m}$  are both plurisubharmonic, and  $0 \leq v'_{t_0,\varepsilon} \circ \Psi_{v,m} \leq 1$ , we have

$$\begin{aligned} & \sum_{1 \leq j, k \leq n} [(1 - v'_{t_0,\varepsilon} \circ \Psi_{v,m}) \partial_j \bar{\partial}_k (\Psi_{v,m} + \varphi_{v,m}) \\ & + (v'_{t_0,\varepsilon} \circ \Psi_{v,m}) \partial_j \bar{\partial}_k (\varphi_{v,m} + (1 + \delta)\Psi_{v,m})] \alpha_I^j \bar{\alpha}^{\bar{I}k} \geq 0, \end{aligned} \quad (3.6)$$

which means

$$\sum_{1 \leq j, k \leq n} \left( \frac{1}{\delta} \partial_j \bar{\partial}_k (\Psi_{v,m} + \varphi_{v,m}) + (v'_{t_0,\varepsilon} \circ \Psi_{v,m}) \partial_j \bar{\partial}_k \Psi_{v,m} \right) \alpha_I^j \bar{\alpha}^{\bar{I}k} \geq 0. \quad (3.7)$$

Because of  $v'_{t_0,\varepsilon} \geq 0$  and  $s' - su' = 1$ , by inequalities (3.2) and (3.7), we have

$$\int_{D_v} (\eta + g^{-1}) |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} dV_M \geq \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\alpha_\perp (\bar{\partial} \Psi_{v,m})^\#|^2 e^{-\Phi} dV_M. \quad (3.8)$$

Let  $\lambda = \bar{\partial}[(1 - v'_{t_0,\varepsilon}(\Psi_{v,m}))\tilde{F}]$ . By the definition of contraction, Cauchy-Schwarz inequality and inequality (3.8), we can see that

$$\begin{aligned} |(\lambda, \alpha)_{D_v, \Phi}|^2 &= |((v''_{t_0,\varepsilon} \circ \Psi_{v,m}) \bar{\partial} \Psi_{v,m} \wedge \tilde{F}, \alpha)_{D_v, \Phi}|^2 \\ &= |((v''_{t_0,\varepsilon} \circ \Psi_{v,m}) \tilde{F}, \alpha_\perp (\bar{\partial} \Psi_{v,m})^\#)_{D_v, \Phi}|^2 \\ &\leq \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\tilde{F}|^2 e^{-\Phi} dV_M \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\alpha_\perp (\bar{\partial} \Psi_{v,m})^\#|^2 e^{-\Phi} dV_M \\ &\leq \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\tilde{F}|^2 e^{-\Phi} dV_M \int_{D_v} (\eta + g^{-1}) |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} dV_M. \end{aligned} \quad (3.9)$$

By Lemma 2.2, we have  $(n, 0)$ -form  $u_{v,t_0,m,\varepsilon}$  on  $D_v$  satisfying  $\bar{\partial} u_{v,t_0,m,\varepsilon} = \lambda$ , such that

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Phi} dV_M \leq \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\tilde{F}|^2 e^{-\Phi} dV_M. \quad (3.10)$$

Denote  $\mu_1 = e^{v_{t_0,\varepsilon} \circ \Psi_{v,m}}$ ,  $\mu = \mu_1 e^\phi$ . Assume that we can choose  $\eta$  and  $\phi$  such that  $\mu \leq C(\eta + g^{-1})^{-1}$ , where  $C$  is just the constant in Theorem 1.7.

Note that  $v_{t_0,\varepsilon}(\Psi_{v,m}) \geq \Psi_{v,m}$ , then we obtain

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_{v,m}} dV_M \leq \int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 \mu_1 e^{\phi} e^{-\varphi_{v,m} - \Psi_{v,m} - \phi} dV_M. \quad (3.11)$$

By inequalities (3.10) and (3.11), we can obtain that

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_{v,m}} dV_M \leq C \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\tilde{F}|^2 e^{-\Phi} dV_M,$$

under the assumption  $\mu \leq C(\eta + g^{-1})^{-1}$ . As  $\varphi$  is continuous,  $\phi$  and  $(v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\tilde{F}|^2 e^{-\Psi_{v,m}}$  are uniformly bounded on  $\overline{D_v}$  independent of  $m$  (because of  $\text{Supp}(v_{t_0,\varepsilon}) \subset \subset (-t_0 - 1, -t_0)$ ), it is clear that  $\int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_{v,m}) |\tilde{F}|^2 e^{-\Phi} dV_M$  are uniformly bounded independent of  $m$ , for any given  $v$ ,  $t_0$  and  $\varepsilon$ . By weakly compactness of the unit ball of  $L^2_\varphi(D_v)$  and dominated convergence theorem, we can see that the weak limit of some weakly convergent subsequence of  $\{u_{v,t_0,m,\varepsilon}\}_m$  when  $m \rightarrow +\infty$  gives an  $(n, 0)$ -form  $u_{v,t_0,\varepsilon}$  on  $D_v$  satisfying

$$\int_{D_v} |u_{v,t_0,\varepsilon}|^2 e^{-\varphi} dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M, \quad (3.12)$$

where  $A_{t_0} := \sup_{t \geq t_0} \{u(t)\}$ . As  $\lim_{t \rightarrow +\infty} u(t) = -\log(1 + \frac{1}{\delta})$ , it is clear that  $\lim_{t_0 \rightarrow \infty} \frac{1}{e^{A_{t_0}}} = 1 + \frac{1}{\delta}$ .

As  $\varphi_{v,m} + \Psi_{v,m}$  is decreasing to  $\varphi + \Psi$  on  $M$ , then for any given  $t_0$  there exists  $m_0$  and a neighborhood  $U_0$  of  $\{\Psi = -\infty\} \cap \overline{D_v}$  in  $M$ , such that for any  $m \geq m_0$  and  $\varepsilon$ ,  $v''_{t_0,\varepsilon} \circ \Psi_{v,m}|_{U_0} = 0$ . Now we will show that  $u_{v,t_0,\varepsilon}$  satisfies  $u_{v,t_0,\varepsilon}|_S = 0$ .

As  $v''_{t_0,\varepsilon} \circ \Psi_{v,m}|_{U_0} = 0$  for  $m > m_0$ , it is clear that  $\bar{\partial}u_{v,t_0,m,\varepsilon}|_{U_0} = 0$ , which implies that  $u_{v,t_0,\varepsilon}|_{U_0}$  is holomorphic.

Note that  $\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_{v,m}} dV_M$  have uniform bound independent of  $m$ . Then we can choose a subsequence with respect to  $m$  from the chosen weakly convergent subsequence of  $u_{v,t_0,m,\varepsilon}$ , such that the subsequence is uniformly convergent on any compact subset of  $U_0$ , and we still denote the subsequence by  $u_{v,t_0,m,\varepsilon}$  without ambiguity.

By inequality (3.10), we can see that  $\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\varphi_{v,m} - \phi - \Psi_{v,m}} dV_M$  are uniformly bounded independent of  $m$ .

Note  $(\eta + g^{-1})^{-1} = (s(-v_{t_0,\varepsilon} \circ \Psi_{v,m}) + \frac{s'^2}{u''_{s-s''}} \circ (-v_{t_0,\varepsilon} \circ \Psi_{v,m}))^{-1}$  and  $e^{-\varphi_{v,m} - \phi} = e^{-\varphi_{v,m} - u(-v_{t_0,\varepsilon} \circ \Psi_{v,m})}$  have positive uniform bound independent of  $m$ . Then  $\int_{K_0} |u_{v,t_0,m,\varepsilon}|^2 e^{-\Psi_{v,m}} dV_M$  have uniform bound independent of  $m$  for any compact set  $K_0 \subset \subset U_0 \cap D_v$ .

As  $\Psi_{v,m'} + \varphi_{v,m'} \leq \Psi_{v,m} + \varphi_{v,m}$ , where  $m' \geq m$ , we have

$$|u_{v,t_0,m',\varepsilon}|^2 e^{-\Psi_{v,m} - \varphi_{v,m}} \leq |u_{v,t_0,m',\varepsilon}|^2 e^{-\Psi_{v,m'} - \varphi_{v,m'}}.$$

Note that  $\varphi_{v,m}$  have uniform bound independent of  $m$ , then  $\int_{K_0} |u_{v,t_0,m',\varepsilon}|^2 e^{-\Psi_{v,m}} dV_M$  have uniform bound independent of  $m$  and  $m'$ , for any compact set  $K_0 \subset \subset U_0 \cap D_v$ .

It is clear that  $\int_{K_0} |u_{v,t_0,\varepsilon}|^2 e^{-\Psi_{v,m}} dV_M$  have uniformly bound independent of  $m$ , for any compact set  $K_0 \subset \subset U_0 \cap D_v$ . Then we have  $\int_{K_0} |u_{v,t_0,\varepsilon}|^2 e^{-\Psi_v} dV_M < \infty$ , for any compact set  $K_0 \subset \subset U_0 \cap D_v$ .

Because of  $\Psi \in \Delta_{\varphi,\delta}(S)$  and  $\Psi_v = \Psi|_{D_v}$ , and by condition (2) in the definition of  $\#(S)$ , we see that  $e^{-\Psi}$  is disintegrable near  $S$ . Then it is clear that  $u_{v,t_0,\varepsilon}$  satisfies  $u_{v,t_0,\varepsilon}|_S = 0$ .

Let  $F_{v,t_0,\varepsilon} := (1 - v'_{t_0,\varepsilon} \circ \Psi_v) \tilde{F} - u_{v,t_0,\varepsilon}$ . By inequality (3.12) and  $u_{v,t_0,\varepsilon}|_S = 0$ , we can see that  $F_{v,t_0,\varepsilon}$  is a holomorphic  $(n, 0)$ -form on  $D_v$  satisfying  $F_{v,t_0,\varepsilon}|_S = \tilde{F}|_S$  and

$$\int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \Psi_v) \tilde{F}|^2 e^{-\varphi} dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M. \quad (3.13)$$

Given  $t_0$  and  $D_v$ , it is clear that  $(v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v}$  have uniform bound on  $D_v$  independent of  $\varepsilon$ . Then  $\int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \Psi_v) \tilde{F}|^2 e^{-\varphi} dV_M$  and  $\int_{D_v} v''_{t_0,\varepsilon} \circ \Psi_v |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M$  have uniform bound independent of  $\varepsilon$ , for any given  $t_0$  and  $D_v$ .

As  $\bar{\partial}F_{v,t_0,\varepsilon} = 0$  and weakly compactness of the unit ball of  $L^2_\varphi(D_v)$ , we see that the weak limit of some weakly convergent subsequence of  $\{F_{v,t_0,\varepsilon}\}_\varepsilon$  when  $\varepsilon \rightarrow 0$  gives us a holomorphic  $(n, 0)$ -form  $F_{v,t_0}$  on  $D_v$  satisfying  $F_{v,t_0}|_S = \tilde{F}|_S$ .

Note that one can also choose a subsequence of the weakly convergent subsequence of  $\{F_{v,t_0,\varepsilon}\}_\varepsilon$ , such that the chosen sequence is uniformly convergent on any compact subset of  $D_v$ , still denoted by  $\{F_{v,t_0,\varepsilon}\}_\varepsilon$  without ambiguity.

For any compact subset  $K$  in  $D_v$ , it is clear that  $F_{v,t_0,\varepsilon}$ ,  $(1 - v'_{t_0,\varepsilon} \circ \Psi_v) \tilde{F}$  and  $(v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v}$  have uniform bound on  $K$  independent of  $\varepsilon$ .

By using dominated convergence theorem on any compact subset  $K$  of  $D_v$  and inequality (3.13), we have

$$\int_K |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|^2 e^{-\varphi} dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M, \quad (3.14)$$

which implies

$$\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|^2 e^{-\varphi} dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M. \quad (3.15)$$

By the definition of  $dV_M[\Psi]$  and  $\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi] < \infty$ , we have

$$\begin{aligned} & \limsup_{t_0 \rightarrow +\infty} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M \\ & \leq \limsup_{t_0 \rightarrow +\infty} \int_M 1_{\overline{D}_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi) |\tilde{F}|^2 e^{-\varphi - \Psi} dV_M \\ & \leq \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} 1_{\overline{D}_v} |f|^2 e^{-\varphi} dV_M[\Psi] \leq \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi] < \infty. \end{aligned} \quad (3.16)$$

Then  $\int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M$  have uniform bound independent of  $t_0$ , for any given  $D_v$ , and

$$\begin{aligned} & \limsup_{t_0 \rightarrow +\infty} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v} dV_M \\ & \leq \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi] < \infty. \end{aligned} \quad (3.17)$$

It is clear that  $\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|^2 e^{-\varphi} dV_M$  have uniform bound independent of  $t_0$ , for any given  $D_v$ .

As  $\int_{D_v} |(1 - b_{t_0}(\Psi_v)) \tilde{F}|^2 e^{-\varphi} dV_M$  have uniform bound independent of  $t_0$ , by inequality (3.15) and

$$\begin{aligned} & \left( \int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|^2 e^{-\varphi} dV_M \right)^{\frac{1}{2}} + \left( \int_{D_v} |(1 - b_{t_0}(\Psi_v)) \tilde{F}|^2 e^{-\varphi} dV_M \right)^{\frac{1}{2}} \\ & \geq \left( \int_{D_v} |F_{v,t_0}|^2 e^{-\varphi} dV_M \right)^{\frac{1}{2}}, \end{aligned} \quad (3.18)$$

we can obtain that  $\int_{D_v} |F_{v,t_0}|^2 e^{-\varphi} dV_M$  have uniform bound independent of  $t_0$ .

By  $\bar{\partial} F_{v,t_0} = 0$  and weakly compactness of the unit ball of  $L^2_{\varphi}(D_v)$ , one can see that the weak limit of some weakly convergent subsequence of  $\{F_{v,t_0}\}_{t_0}$  when  $t_0 \rightarrow +\infty$  gives us a holomorphic  $(n, 0)$ -form  $F_v$  on  $D_v$  satisfying  $F_v|_S = \tilde{F}|_S$ .

Note that we can also choose a subsequence of the weakly convergent subsequence of  $\{F_{v,t_0}\}_{t_0}$ , such that the chosen sequence is uniformly convergent on any compact subset of  $D_v$ , denoted again by  $\{F_{v,t_0}\}_{t_0}$  without ambiguity.

For any compact subset  $K$  on  $D_v$ , it is clear that both  $F_{v,t_0}$  and  $(1 - b_{t_0} \circ \Psi_v) \tilde{F}$  have uniform bound on  $K$  independent of  $t_0$ .

By inequalities (3.15), (3.17) and dominated convergence theorem on any compact subset  $K$  of  $D_v$ , we have

$$\int_{D_v} 1_K |F_v|^2 e^{-\varphi} dV_M \leq \frac{C}{e^{A_{t_0}}} \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi], \quad (3.19)$$

which implies

$$\int_{D_v} |F_v|^2 e^{-\varphi} dV_M \leq \frac{C}{e^{A_{t_0}}} \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi]. \quad (3.20)$$

Note that the Lebesgue measure of  $\{\Psi = -\infty\}$  is zero.

Now it suffices to find  $\eta$  and  $\phi$  such that  $(\eta + g^{-1}) \leq C e^{-\Psi_{v,\delta}} e^{-\phi} = C \mu^{-1}$  on  $D_v$ . As  $\eta = s(-v_{t_0,\varepsilon} \circ \Psi_{v,\delta})$  and  $\phi = u(-v_{t_0,\varepsilon} \circ \Psi_{v,\delta})$ , we have  $(\eta + g^{-1}) e^{v_{t_0,\varepsilon} \circ \Psi_{v,\delta}} e^{\phi} = (s + \frac{s'^2}{u''s - s''}) e^{-t} e^u \circ (-v_{t_0,\varepsilon} \circ \Psi_{v,\delta})$ .

We are naturally led to obtain the following system of ODEs:

$$\begin{aligned} (1) \quad & \left( s + \frac{s'^2}{u''s - s''} \right) e^{u-t} = C, \\ (2) \quad & s' - su' = 1, \end{aligned} \quad (3.21)$$

where  $t \in [0, +\infty)$ , and  $C = 1$ .

One can solve the ODE (3.21) (for details, see the following remark) to get  $u = -\log(1 + \frac{1}{\delta} - e^{-t})$  and  $s = \frac{(1+\frac{1}{\delta})t + \frac{1}{\delta}(1+\frac{1}{\delta})}{1+\frac{1}{\delta}-e^{-t}} - 1$ .

One may check that  $s \in C^\infty((0, +\infty))$  satisfies  $s \geq \frac{1}{\delta}$ ,  $\lim_{t \rightarrow +\infty} u(t) = -\log(1 + \frac{1}{\delta})$  and  $u \in C^\infty((0, +\infty))$  satisfies  $u''s - s'' > 0$ .

**Remark 3.1.** Now we solve the equation (3.21):

By (2) of equation (3.21), we have  $su'' - s'' = -s'u'$ . Then (1) of equation (3.21) can be changed into

$$\left(s - \frac{s'}{u'}\right)e^{u-t} = C,$$

which is

$$\frac{su' - s'}{u'}e^{u-t} = C.$$

By (2) of equation (3.21), we have

$$C = \frac{su' - s'}{u'}e^{u-t} = \frac{-1}{u'}e^{u-t},$$

which is

$$\frac{de^{-u}}{dt} = -u'e^{-u} = \frac{e^{-t}}{C}.$$

Note that (2) of equation (3.21) is equivalent to  $\frac{d(se^{-u})}{dt} = e^{-u}$ . As  $s \geq 0$ , we obtain the solution

$$\begin{cases} u = -\log(a - e^{-t}), \\ s = \frac{at + e^{-t} + b}{a - e^{-t}}, \end{cases}$$

when  $C = 1$ , where  $a \geq 1$  and  $b \geq -1$ .

As  $\lim_{t \rightarrow +\infty} u(t) = -\log(1 + \frac{1}{\delta})$ , we have  $a = (1 + \frac{1}{\delta})$ .

By  $su'' - s'' = -s'u'$ , it is clear that  $u''s - s'' > 0$  is equivalent to  $s' > 0$ , which implies  $b \leq a^2 - 2a$  (by considering the limit at 0 and  $\infty$  of  $s'$ ).

As  $s \geq \frac{1}{\delta}$ , we have  $b = a^2 - 2a = \frac{1}{\delta^2} - 1$ .

Define  $F_v = 0$  on  $M \setminus D_v$ . As  $\lim_{t_0 \rightarrow \infty} A_{t_0} = -\log(1 + \frac{1}{\delta})$ , then the weak limit of some weakly convergent subsequence of  $\{F_v\}_{v=1}^\infty$  gives us a holomorphic  $(n, 0)$ -form  $F$  on  $M$  satisfying  $F|_S = \tilde{F}|_S$ , and

$$c_n \int_M e^{-\varphi} F \wedge \bar{F} = \int_M |F|^2 e^{-\varphi} dV_M \leq C \left(1 + \frac{1}{\delta}\right) \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|^2 e^{-\varphi} dV_M[\Psi],$$

where  $c_k = (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^k$  for  $k \in \mathbb{Z}$ .

In conclusion, we have proved Theorem 1.12 with the constant  $C = 1$ .

### 3.2 Proof of Remark 1.8

Let  $\Delta$  be the unit disc on  $\mathbb{C}$ , with coordinate  $z$ . Let

$$\varphi(z) = (1 + \delta) \max\{\log|z|^2, \log|a|^2\}, \quad \text{and} \quad \Psi(z) = -\max\{\log|z|^2, \log|a|^2\} + \log|z|^2 - \varepsilon,$$

where  $a \in (0, 1)$ ,  $\varepsilon > 0$ .

As  $\varphi$  and  $\varphi + (1 + \delta)\Psi$  are both plurisubharmonic, and

$$\varphi + \Psi = \frac{\delta\varphi + (\varphi + (1 + \delta)\Psi)}{1 + \delta},$$

it is clear that  $\Psi(z) \in \Delta_{\varphi, \delta}(S)$ , where  $S = \{z = 0\}$ .



For any  $f(0) \neq 0$ , it suffices to prove

$$\lim_{a \rightarrow 0} \frac{\min_{F \in \text{Hol}(\Delta)} \int_{\Delta} |F|^2 e^{-\varphi} d\lambda}{a^{-2\delta} e^{\varepsilon} |F(0)|^2} = (1 + \delta^{-1}) \frac{\pi}{e^{\varepsilon}}, \quad (3.22)$$

where  $F(0) = f(0)$ . It is because  $e^{-\varphi} d\lambda[\Psi] = a^{-2\delta} e^{\varepsilon} \delta_0$  (by Lemma 2.7),  $\delta_0$  is the dirac function at 0, when  $\varepsilon$  goes to zero, then we can see that the constant of Theorem 1.7 is optimal.

By Taylor expansion, at  $0 \in \mathbb{C}$ ,  $F(z) = \sum_{k=0}^{\infty} a_k z^k$ , where  $a_k$  are complex constants. Note that  $\int_{\Delta} z^{k_1} \bar{z}^{k_2} e^{-\varphi} d\lambda = 0$  when  $k_1 \neq k_2$ , and  $\int_{\Delta} z^{k_1} \bar{z}^{k_2} e^{-\varphi} d\lambda > 0$  when  $k_1 = k_2$ . It is clear that

$$\min_{F \in \text{Hol}(\Delta)} \int_{\Delta} |F|^2 e^{-\varphi} d\lambda = \int_{\Delta} |F(0)|^2 e^{-\varphi} d\lambda.$$

As

$$\int_{\Delta} e^{-\varphi} d\lambda = \pi \left( \frac{a^{-2\delta} - 1}{\delta} + a^{-2\delta} \right) \quad \text{and} \quad \lim_{a \rightarrow 0} \frac{\frac{a^{-2\delta} - 1}{\delta} + a^{-2\delta}}{a^{-2\delta}} = 1 + \frac{1}{\delta},$$

we can prove the equality (3.22).

In the following part, we will show that for any given planar domain and point in the domain, the constant  $C$  is also optimal.

Let  $\Omega$  be a planar domain in  $\mathbb{C}$ , such that unit disc  $\Delta \subset \Omega$ . Let

$$\varphi_N(z) = (1 + \delta) \max\{\log |z|^2, \log |a|^2, N \log |z|^2\},$$

and

$$\Psi_N(z) = -\max\{\log |z|^2, \log |a|^2, N \log |z|^2\} + \log |z|^2 - \varepsilon,$$

where  $a \in (0, 1)$ ,  $\varepsilon > 0$ ,  $N > 3$ . It is clear that  $\varphi_N(z)|_{\Delta} = \varphi(z)$ , and  $\Psi_N(z)|_{\Delta} = \Psi(z)$ .

By the same arguments as above, we can see that

$$\min_{F \in \text{Hol}(\Omega)} \int_{\Omega} |F|^2 e^{-\varphi_N} d\lambda \geq \int_{\Delta} |F(0)|^2 e^{-\varphi_N} d\lambda.$$

By the similar calculations, we can obtain that for any  $f(0) \neq 0$ ,

$$\lim_{a \rightarrow 0} \frac{\min_{F \in \text{Hol}(\Omega)} \int_{\Omega} |F|^2 e^{-\varphi_N} d\lambda}{a^{-2\delta} e^{\varepsilon} |F(0)|^2} \geq \lim_{a \rightarrow 0} \frac{\min_{F \in \text{Hol}(\Delta)} \int_{\Delta} |F|^2 e^{-\varphi} d\lambda}{a^{-2\delta} e^{\varepsilon} |F(0)|^2} = (1 + \delta^{-1}) \frac{\pi}{e^{\varepsilon}}, \quad (3.23)$$

where  $F(0) = f(0)$ .

Then we have shown that for any given planar domain and point in the domain, the constant  $C$  is also optimal.

### 3.3 Proof of Theorem 1.12

By Remark 1.6, it suffices to prove the case when  $M$  is a Stein manifold.

Since  $M$  is Stein, we can find a sequence of strictly pseudoconvex domains  $\{D_v\}_{v=1}^{\infty}$  with smooth boundaries satisfying  $D_v \subset \subset D_{v+1}$  for all  $v$  and  $\bigcup_{v=1}^{\infty} D_v = M$ .

As  $\Psi < 0$ , by Lemma 2.5, we can choose a sequence of smooth plurisubharmonic functions  $\{\Psi_{v,m}\}_{m=1,2,\dots}$  on  $M$ , such that  $\Psi_{v,m}$  decreasingly converge to  $\Psi$  on  $D_v$  and  $\Psi_{v,m}|_{\overline{D_v}} < 0$ . Denote  $\Psi_v := \Psi|_{D_v}$ .

Just like the arguments in [33] and [29], we can assume that  $\varphi$  is smooth on  $M$  as in Theorem 1.12.

Since  $M$  is Stein, there is a holomorphic section  $\tilde{F}$  of  $K_M$  on  $M$  such that  $\tilde{F}|_S = f$ .

Let  $ds_M^2$  be a Kähler metric on  $M$ , and  $dV_M$  be the volume form with respect to  $ds_M^2$ . As  $\Psi$  is plurisubharmonic, it is clear that for any  $\delta > 0$ , we have  $\Psi \in \Delta_{\varphi,\delta}(S)$ .

When  $\delta$  approaches to infinity, we obtain Theorem 1.12 by Theorem 1.7.

### 3.4 Proof of Corollary 1.14

As  $\kappa_{M/S}$  has extremal property, we have

$$\kappa_{M/S} = \sup_f \frac{f \otimes \bar{f}}{2^{-n} \sum_{k=1}^n \int_{S_{n-k}} \frac{c_n f \wedge \bar{f}}{dV_M} dV_M[G(\cdot, S)]} = \sup_f \frac{f \otimes \bar{f}}{2^{-n} \int_{S_{n-k}} \frac{c_n f \wedge \bar{f}}{dV_M} dV_M[G(\cdot, S)]}, \quad (3.24)$$

for all holomorphic section  $f$  of  $K_X|_{S_{n-k}}$ .

By Lemma 2.8, we have  $G(\cdot, S) \in \Delta(S)$ . Denote  $\Psi := G(\cdot, S)$ .

By Theorem 1.12, we have

$$\begin{aligned} \frac{F \otimes \bar{F}}{2^{-n} c_n \int_M F \wedge \bar{F}} &\geq \frac{f \otimes \bar{f}}{2^{-n} \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{c_n f \wedge \bar{f}}{dV_M} dV_M[G(\cdot, S)]} \\ &= \frac{f \otimes \bar{f}}{2^{-n} \frac{\pi^k}{k!} \int_{S_{n-k}} \frac{c_n f \wedge \bar{f}}{dV_M} dV_M[G(\cdot, S)]}. \end{aligned} \quad (3.25)$$

Then we can obtain that

$$\frac{\pi^k}{k!} \frac{F \otimes \bar{F}}{2^{-n} c_n \int_M F \wedge \bar{F}} \geq \frac{f \otimes \bar{f}}{2^{-n} \int_{S_{n-k}} \frac{c_n f \wedge \bar{f}}{dV_M} dV_M[G(\cdot, S)]},$$

where  $F$  is an extension of  $f$ .

As

$$\kappa_{M/S} = \sup_f \frac{f \otimes \bar{f}}{(2^{-n} \int_{S_{n-k}} \frac{c_n f \wedge \bar{f}}{dV_M} dV_M[(G(\cdot, S))])},$$

we prove the corollary.

### 3.5 Proof of Theorem 1.1

Just like the arguments in [33] and [29], one may assume that  $H = S$  and that both  $\varphi$  and  $\psi$  are smooth on  $X$  as in Theorem 1.1. Let  $\Psi = \log |w^2| + \psi$ . By Lemma 2.7, we have

$$d\lambda_z[\Psi] = d\lambda_z[\log |w^2| + \psi] = e^{-\psi} d\lambda_{z'}. \quad (3.26)$$

By Theorem 1.12, we get the theorem.

### 3.6 Proof of Theorem 1.21

By Remark 1.6, it suffices to prove the theorem for the case when  $M$  is a Stein manifold.

Since  $M$  is a Stein manifold, we can find a sequence of Stein manifolds  $\{D_v\}_{v=1}^\infty$  satisfying  $D_v \subset \subset D_{v+1}$  for all  $v$  and  $\bigcup_{v=1}^\infty D_v = M$ , and  $D_v \setminus S$  are all complete Kähler. Let  $\Psi_v := \Psi|_{D_v}$ .

Since  $M$  is Stein, there is a holomorphic section  $\tilde{F}$  of  $K_M$  on  $M$  such that  $\tilde{F}|_S = f$ . Let  $ds_M^2$  be a Kähler metric on  $M$ , and  $dV_M$  is the volume form with respect to  $ds_M^2$ .

Let  $\{v_{t_0, \varepsilon}\}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{4})}$  be family of smooth increasing convex functions on  $\mathbb{R}$  (also continuous functions on  $\mathbb{R} \cup -\infty$ ) which are the same as in the proof of Theorem 1.7.

Therefore we have

$$v''_{t_0, \varepsilon} = \frac{1}{1-2\varepsilon} 1_{\{-t_0-1+\varepsilon < t < -t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon}, \quad \text{and} \quad v'_{t_0, \varepsilon} = \int_{-\infty}^t \frac{1}{1-2\varepsilon} 1_{\{-t_0-1+\varepsilon < s < -t_0-\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon} ds.$$

Let  $\eta = s(-v_{t_0, m} \circ \Psi)$  and  $\phi = u(-v_{t_0, \varepsilon} \circ \Psi)$ , where  $s \in C^\infty((0, +\infty))$  satisfies  $s \geq \frac{1}{\delta}$ , and  $u \in C^\infty((0, +\infty))$  satisfies  $\lim_{t \rightarrow +\infty} u(t) = -\log(1 + \frac{1}{\delta})$ , such that  $u''s - s'' > 0$ , and  $s' - u's = 1$ . Let  $h' = he^{-\Psi-\phi}$ .

Now let  $\alpha \in \mathcal{D}(X, \Lambda^{n,1} T_{M \setminus S}^* \otimes E)$  with compact support on  $M \setminus S$ . By Lemma 2.3,  $s \geq \frac{1}{\delta}$  and  $\Theta_{he^{-\Psi}} \geq 0$  on  $M \setminus S$ , we get

$$\begin{aligned} & \|(\eta + g^{-1})^{\frac{1}{2}} D''^* \alpha\|_{D_v \setminus S, h'}^2 + \|\eta^{\frac{1}{2}} D'' \alpha\|_{D_v \setminus S, h'}^2 \\ & \geq \langle [\eta \sqrt{-1} \Theta_{h'} - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega] \alpha, \alpha \rangle_{D_v \setminus S, h'} \\ & \geq \left\langle \left[ \eta \sqrt{-1} \partial \bar{\partial} \phi + \frac{1}{\delta} \sqrt{-1} \Theta_{he^{-\Psi}} - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega \right] \alpha, \alpha \right\rangle_{D_v \setminus S, h'}, \end{aligned} \quad (3.27)$$

where  $g$  is a positive continuous function on  $D_v \setminus S$ . We need some calculations to determine  $g$ .

We have

$$\partial \bar{\partial} \eta = -s'(-v_{t_0, \varepsilon} \circ \Psi) \partial \bar{\partial}(v_{t_0, \varepsilon} \circ \Psi) + s''(-v_{t_0, \varepsilon} \circ \Psi) \partial(v_{t_0, \varepsilon} \circ \Psi) \wedge \bar{\partial}(v_{t_0, \varepsilon} \circ \Psi), \quad (3.28)$$

and

$$\partial \bar{\partial} \phi = -u'(-v_{t_0, \varepsilon} \circ \Psi) \partial \bar{\partial}(v_{t_0, \varepsilon} \circ \Psi) + u''(-v_{t_0, \varepsilon} \circ \Psi) \partial(v_{t_0, \varepsilon} \circ \Psi) \wedge \bar{\partial}(v_{t_0, \varepsilon} \circ \Psi). \quad (3.29)$$

Therefore

$$\begin{aligned} & \eta \sqrt{-1} \partial \bar{\partial} \phi - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta \\ & = (s' - su') \sqrt{-1} \partial \bar{\partial}(v_{t_0, \varepsilon} \circ \Psi) + ((u''s - s'') - gs'^2) \sqrt{-1} \partial(v_{t_0, \varepsilon} \circ \Psi) \wedge \bar{\partial}(v_{t_0, \varepsilon} \circ \Psi) \\ & = (s' - su')((v'_{t_0, \varepsilon} \circ \Psi) \sqrt{-1} \partial \bar{\partial} \Psi + (v''_{t_0, \varepsilon} \circ \Psi) \sqrt{-1} \partial(\Psi) \wedge \bar{\partial}(\Psi)) \\ & \quad + ((u''s - s'') - gs'^2) \sqrt{-1} \partial(v_{t_0, \varepsilon} \circ \Psi) \wedge \bar{\partial}(v_{t_0, \varepsilon} \circ \Psi). \end{aligned} \quad (3.30)$$

We omit composite item  $(-v_{t_0, \varepsilon} \circ \Psi)$  after  $s' - su'$  and  $(u''s - s'') - gs'^2$  in the above equalities.

Denote  $g = \frac{u''s - s''}{s'^2} \circ (-v_{t_0, \varepsilon} \circ \Psi)$ . We have  $\eta + g^{-1} = (s + \frac{s'^2}{u''s - s''}) \circ (-v_{t_0, \varepsilon} \circ \Psi)$ . Since  $\Theta_{he^{-\Psi}} \geq 0$  and  $\Theta_{he^{-(1+\delta)\Psi}} \geq 0$  on  $M \setminus S$  and  $0 \leq v'_{t_0, \varepsilon} \circ \Psi \leq 1$ , we have

$$(1 - v'_{t_0, \varepsilon} \circ \Psi) \Theta_{he^{-\Psi}} + (v'_{t_0, \varepsilon} \circ \Psi) \Theta_{he^{-(1+\delta)\Psi}} \geq 0 \quad (3.31)$$

on  $M \setminus S$ , which means

$$\frac{1}{\delta} \Theta_{he^{-\Psi}} + (v'_{t_0, \varepsilon} \circ \Psi) \partial \bar{\partial} \Psi \geq 0 \quad (3.32)$$

on  $M \setminus S$ .

As  $v'_{t_0, \varepsilon} \geq 0$  and  $s' - su' = 1$ , by equalities (3.27), (3.30) and inequality (3.32), we have

$$\begin{aligned} \langle B\alpha, \alpha \rangle_{h'} & = \langle [\eta \sqrt{-1} \Theta_{h'} - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega] \alpha, \alpha \rangle_{h'} \\ & \geq \langle [(v''_{t_0, \varepsilon} \circ \Psi) \sqrt{-1} \partial \bar{\partial} \Psi + (v''_{t_0, \varepsilon} \circ \Psi) \sqrt{-1} \partial(\Psi) \wedge \bar{\partial}(\Psi), \Lambda_\omega] \alpha, \alpha \rangle_{h'} = \langle (v''_{t_0, \varepsilon} \circ \Psi) \bar{\partial} \Psi \wedge (\alpha_\perp(\bar{\partial} \Psi)^\sharp), \alpha \rangle_{h'}. \end{aligned} \quad (3.33)$$

By the definition of contraction, Cauchy-Schwarz inequality and the inequality (3.33), we have

$$\begin{aligned} |\langle (v''_{t_0, \varepsilon} \circ \Psi) \bar{\partial} \Psi \wedge u, v \rangle_{h'}|^2 & = |\langle (v''_{t_0, \varepsilon} \circ \Psi) u, v_\perp(\bar{\partial} \Psi)^\sharp \rangle_{h'}|^2 \\ & \leq \langle (v''_{t_0, \varepsilon} \circ \Psi) u, u \rangle_{h'} \langle (v''_{t_0, \varepsilon} \circ \Psi) |v_\perp(\bar{\partial} \Psi)^\sharp|_{h'}^2 \\ & = \langle (v''_{t_0, \varepsilon} \circ \Psi) u, u \rangle_{h'} \langle (v''_{t_0, \varepsilon} \circ \Psi) \bar{\partial} \Psi \wedge (v_\perp(\bar{\partial} \Psi)^\sharp), v \rangle_{h'} \\ & \leq \langle (v''_{t_0, \varepsilon} \circ \Psi) u, u \rangle_{h'} \langle Bv, v \rangle_{h'}, \end{aligned} \quad (3.34)$$

for any  $(n, q)$ -form  $u$  and  $(n, q+1)$ -form  $v$ .

Let  $\lambda = \bar{\partial}[(1 - v'_{t_0, \varepsilon}(\Psi))\tilde{F}]$ ,  $u = \tilde{F}$ , and  $v = B^{-1}\bar{\partial}\Psi \wedge \tilde{F}$ , we have

$$\langle B^{-1}\lambda, \lambda \rangle_{h'} \leq (v''_{t_0, \varepsilon} \circ \Psi) |\tilde{F}|_{h'}^2.$$

Then it is clear that

$$\int_{D_v \setminus S} \langle B^{-1}\lambda, \lambda \rangle_{h'} dV_M \leq \int_{D_v \setminus S} (v''_{t_0, \varepsilon} \circ \Psi) |\tilde{F}|_{h'}^2 dV_M.$$

By Lemma 2.4, we get  $u_{v,t_0,\varepsilon}$  on  $D_v \setminus S$  which is an  $(n, 0)$ -form with values in  $E$  satisfying  $\bar{\partial}u_{v,t_0,\varepsilon} = \lambda$ , such that

$$\int_{D_v \setminus S} |u_{v,t_0,\varepsilon}|_{h'}^2 (\eta + g^{-1})^{-1} dV_M \leq \int_{D_v \setminus S} (v''_{t_0,\varepsilon} \circ \Psi) |\tilde{F}|_{h'}^2 dV_M. \quad (3.35)$$

Denote  $\mu_1 = e^{v_{t_0,\varepsilon} \circ \Psi}$  and  $\mu = \mu_1 e^\phi$ . Assume one can choose  $\eta$  and  $\phi$  such that  $\mu \leq C(\eta + g^{-1})^{-1}$ , where  $C$  is just the constant in Theorem 1.21.

Note that  $v_{t_0,\varepsilon}(\Psi) \geq \Psi$ , then we obtain

$$\int_{D_v \setminus S} |u_{v,t_0,\varepsilon}|_h^2 dV_M \leq \int_{D_v \setminus S} |u_{v,t_0,\varepsilon}|_{h'}^2 \mu_1 e^\phi dV_M. \quad (3.36)$$

By inequalities (3.35) and (3.36), we have

$$\int_{D_v \setminus S} |u_{v,t_0,\varepsilon}|_h^2 dV_M \leq C \int_{D_v \setminus S} (v''_{t_0,\varepsilon} \circ \Psi) |\tilde{F}|_{h'}^2 dV_M,$$

under the assumption  $\mu \leq C(\eta + g^{-1})^{-1}$ .

For any given  $t_0$  there exist  $m_0$  and a neighborhood  $U_0$  of  $\{\Psi = -\infty\} \cap \overline{D_v}$  on  $M$ , such that for any  $\varepsilon$ ,  $v''_{t_0,\varepsilon} \circ \Psi_{v,m}|_{U_0} = 0$ , we have  $\bar{\partial}u_{v,t_0,\varepsilon}|_{U_0 \setminus S} = 0$ .

Note that  $u_{v,t_0,\varepsilon}$  is locally  $L^2$  integrable along  $S$ , we have  $u_{v,t_0,\varepsilon}$  can be extended to  $U_0$  as a holomorphic function, which is denoted by  $\tilde{u}_{v,t_0,\varepsilon}$ .

As  $\Psi \in \Delta_{h,\delta}$ , we see that  $e^{-\Psi}$  is disintegrable near  $S$ . Then it is clear that  $\tilde{u}_{v,t_0,\varepsilon}$  satisfies  $\tilde{u}_{v,t_0,\varepsilon}|_S = 0$ , and

$$\int_{D_v} |\tilde{u}_{v,t_0,\varepsilon}|_h^2 dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi}}^2 dV_M, \quad (3.37)$$

where  $A_{t_0} := \sup_{t \geq t_0} \{u(t)\}$ .

As  $\lim_{t \rightarrow +\infty} u(t) = -\log(1 + \frac{1}{\delta})$ , it is clear that  $\lim_{t_0 \rightarrow \infty} \frac{1}{e^{A_{t_0}}} = 1 + \frac{1}{\delta}$ .

Let  $F_{v,t_0,\varepsilon} := (1 - v'_{t_0,\varepsilon} \circ \Psi_v) \tilde{F} - \tilde{u}_{v,t_0,\varepsilon}$ . By  $\tilde{u}_{v,t_0,\varepsilon}|_S = 0$ , we have that  $F_{v,t_0,\varepsilon}$  is a holomorphic  $(n, 0)$ -form on  $D_v$  satisfying  $F_{v,t_0,\varepsilon}|_S = \tilde{F}|_S$  and inequality (3.37) is reformulated as follows:

$$\int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \Psi) \tilde{F}|_h^2 dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M. \quad (3.38)$$

Given  $t_0$  and  $D_v$ , it is clear that  $(v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi_v}}^2$  have uniform bound on  $D_v$  independent of  $\varepsilon$ .

Then  $\int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \Psi_v) \tilde{F}|_h^2 dV_M$  and  $\int_{D_v} v''_{t_0,\varepsilon} \circ \Psi_v |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M$  have uniform bound independent of  $\varepsilon$ , for any given  $t_0$  and  $D_v$ .

By  $\bar{\partial}F_{v,t_0,\varepsilon} = 0$  and weakly compactness of the unit ball of  $L^2_\varphi(D_v)$ , we see that the weak limit of some weakly convergent subsequence of  $\{F_{v,t_0,\varepsilon}\}_\varepsilon$  when  $\varepsilon \rightarrow 0$  gives us a holomorphic  $(n, 0)$ -form with values in  $E$ , which is denoted by  $F_{v,t_0}$  on  $D_v$  and satisfies  $F_{v,t_0}|_S = \tilde{F}|_S$ .

Note that we can also choose a subsequence of the weakly convergent subsequence of  $\{F_{v,t_0,\varepsilon}\}_\varepsilon$ , such that the chosen sequence is uniformly convergent on any compact subset of  $D_v$ , denoted again by  $\{F_{v,t_0,\varepsilon}\}_\varepsilon$  without ambiguity.

For any compact subset  $K$  on  $D_v$ , it is clear that  $F_{v,t_0,\varepsilon}$ ,  $(1 - v'_{t_0,\varepsilon} \circ \Psi_v) \tilde{F}$  and  $(v''_{t_0,\varepsilon} \circ \Psi_v) |\tilde{F}|^2 e^{-\varphi - \Psi_v}$  have uniform bounds on  $K$  independent of  $\varepsilon$ .

By using dominated convergence theorem on any compact subset  $K$  of  $D_v$  and inequality (3.38), we have

$$\int_K |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|_h^2 dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M, \quad (3.39)$$

which implies

$$\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|_h^2 dV_M \leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M. \quad (3.40)$$

By the definition of  $dV_M[\Psi]$  and since  $\sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty$ , we have

$$\begin{aligned} & \limsup_{t_0 \rightarrow +\infty} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M \\ & \leq \limsup_{t_0 \rightarrow +\infty} \int_M 1_{\overline{D_v}} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi) |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M \\ & \leq \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} 1_{\overline{D_v}} |f|_h^2 dV_M[\Psi] \leq \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty. \end{aligned} \quad (3.41)$$

Then  $\int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M$  have uniform bound independent of  $t_0$ , for any given  $D_v$ , and

$$\limsup_{t_0 \rightarrow +\infty} \int_{D_v} (1_{\{-t_0-1 < t < -t_0\}} \circ \Psi_v) |\tilde{F}|_{he^{-\Psi_v}}^2 dV_M \leq \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi] < \infty. \quad (3.42)$$

It is clear that  $\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|_h^2 dV_M$  have uniform bound independent of  $t_0$ , for any given  $D_v$ .

As  $\int_{D_v} |(1 - b_{t_0}(\Psi_v)) \tilde{F}|_h^2 dV_M$  have uniform bound independent of  $t_0$ , by inequality (3.40) and

$$\left( \int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi_v)) \tilde{F}|_h^2 dV_M \right)^{\frac{1}{2}} + \left( \int_{D_v} |(1 - b_{t_0}(\Psi_v)) \tilde{F}|_h^2 dV_M \right)^{\frac{1}{2}} \geq \left( \int_{D_v} |F_{v,t_0}|_h^2 dV_M \right)^{\frac{1}{2}}, \quad (3.43)$$

we can obtain that  $\int_{D_v} |F_{v,t_0}|_h^2 dV_M$  have uniform bound independent of  $t_0$ .

Because of  $\bar{\partial} F_{v,t_0} = 0$  and weakly compactness of the unit ball of  $L^2_\varphi(D_v)$ , we see that the weak limit of some weakly convergent subsequence of  $\{F_{v,t_0}\}_{t_0}$  when  $t_0 \rightarrow +\infty$  gives us a holomorphic  $(n, 0)$ -form with values in  $E$ , which is denoted by  $F_v$  on  $D_v$  and satisfies  $F_v|_S = \tilde{F}|_S$ .

Note that we can also choose a subsequence of the weakly convergent subsequence of  $\{F_{v,t_0}\}_{t_0}$ , such that the chosen sequence is uniformly convergent on any compact subset of  $D_v$ , denoted by  $\{F_{v,t_0}\}_{t_0}$  without ambiguity.

For any compact subset  $K$  on  $D_v$ , it is clear that both of  $F_{v,t_0}$  and  $(1 - b_{t_0} \circ \Psi_v) \tilde{F}$  have uniform bound on  $K$  independent of  $t_0$ .

By inequalities (3.40), (3.42) and dominated convergence theorem on any compact subset  $K$  of  $D_v$ , we have

$$\int_{D_v} 1_K |F_v|_h^2 dV_M \leq \frac{C}{e^{A_{t_0}}} \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi], \quad (3.44)$$

which implies

$$\int_{D_v} |F_v|_h^2 dV_M \leq \frac{C}{e^{A_{t_0}}} \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi]. \quad (3.45)$$

Note that the Lebesgue measure of  $\{\Psi = -\infty\}$  is zero.

It suffices to find  $\eta$  and  $\phi$  such that

$$(\eta + g^{-1}) \leq C e^{-\Psi_v} e^{-\phi} = C \mu^{-1} \quad \text{on } D_v.$$

As  $\eta = s(-v_{t_0,\varepsilon} \circ \Psi_v)$  and  $\phi = u(-v_{t_0,\varepsilon} \circ \Psi_v)$ , we have

$$(\eta + g^{-1}) e^{v_{t_0,\varepsilon} \circ \Psi_v} e^{\phi} = \left( s + \frac{s'^2}{u''s - s''} \right) e^{-t} e^u \circ (-v_{t_0,\varepsilon} \circ \Psi_v).$$

We are naturally led to obtain the following system of ODEs:

$$\begin{aligned} (1) \quad & \left( s + \frac{s'^2}{u''s - s''} \right) e^{u-t} = C, \\ (2) \quad & s' - su' = 1, \end{aligned} \quad (3.46)$$

where  $t \in [0, +\infty)$ , and  $C = 1$ .

One can solve the ODEs by the same method as in Remark 3.21 and get

$$u = -\log\left(1 + \frac{1}{\delta} - e^{-t}\right) \quad \text{and} \quad s = \frac{(1 + \frac{1}{\delta})t + \frac{1}{\delta}(1 + \frac{1}{\delta})}{1 + \frac{1}{\delta} - e^{-t}} - 1,$$

which satisfy the ODEs (3.46).

One may check that  $s \in C^\infty((0, +\infty))$  satisfies  $s \geq \frac{1}{\delta}$ ,  $\lim_{t \rightarrow +\infty} u(t) = -\log(1 + \frac{1}{\delta})$  and  $u \in C^\infty((0, +\infty))$  satisfies  $u''s - s'' > 0$ .

Define  $F_v = 0$  on  $M \setminus D_v$ . As  $\lim_{t_0 \rightarrow \infty} A_{t_0} = -\log(1 + \frac{1}{\delta})$ , then the weak limit of some weakly convergent subsequence of  $\{F_v\}_{v=1}^\infty$  gives us a holomorphic section  $F$  of  $K_M \otimes E$  on  $M$  satisfying  $F|_S = \tilde{F}|_S$ , and

$$\int_M |F|_h^2 dV_M \leq C \left(1 + \frac{1}{\delta}\right) \sum_{k=1}^n \frac{\pi^k}{k!} \int_{S_{n-k}} |f|_h^2 dV_M[\Psi],$$

where  $c_k = (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^k$  for  $k \in \mathbb{Z}$ .

In conclusion, we have proved Theorem 1.21 with the constant  $C = 1$ .

## 4 Bergman kernel and logarithmic capacity on Riemann surfaces

In this section, we show some relationships between Bergman kernel and logarithmic capacity on compact and open Riemann surfaces.

### 4.1 Proof of Theorem 1.10

Let  $e^{2\varphi}|dz|^2$  be the Poincaré metric on  $X$ , it is clear that  $\omega = e^{2\varphi}dz \wedge d\bar{z}$ . Then  $e^{-2\varphi}$  is a Hermitian metric of  $K_X$  on  $X$ , denoted by  $h$ . It is known that

$$c_1(K_X) = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial}(2\varphi) = b\omega,$$

where  $b$  is a positive constant, and

$$\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} g(\cdot, q)|_{X \setminus q} = -a\omega,$$

where  $a$  is a positive constant.

Note that

$$\int_X \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} g(\cdot, q) = 0 \quad \text{and} \quad \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} g(\cdot, q) = [\{q\}] - a\omega,$$

then we have

$$\int_X a\omega = 1.$$

Note that

$$\int_X b\omega = \int_X c_1(K_X) = 2(g-1),$$

then it is clear that

$$\frac{2a}{b} = \frac{1}{g-1}.$$

Let  $\Psi = 2g(\cdot, p)$ . We choose

$$\delta = (g-1)(m-1) - 1,$$

then it is clear that

$$\Psi \in \Delta_{h^m, \delta} \quad \text{and} \quad 1 + \frac{1}{\delta} = 1 + \frac{1}{(g-1)(m-1) - 1}.$$

Without loss of generality, we can assume that  $p$  is  $o \in \Delta$ ,  $z \in \Delta$  is the local coordinate near  $p$ . Then by the definition of  $c_X(p)$ , the relation between Euclidean distance and Poincaré distance near  $o \in \Delta$ , and Lemma 2.7, we have

$$\begin{aligned}
 \omega[\Psi] &= e^{2\varphi} dz \wedge d\bar{z} [2 \log \text{dist}_\omega(\cdot, p) + 2 \log c_X(p)] \\
 &= e^{2\varphi} dz \wedge d\bar{z} [2 \log \text{dist}_\omega(\cdot, p)] c_X^{-2}(p) \\
 &= 2e^{2\varphi} d\lambda_z [2 \log |z - p| + 2\varphi(p)] c_X^{-2}(p) \\
 &= 2e^{2\varphi} d\lambda_z [2 \log |z - p|] e^{-2\varphi(p)} c_X^{-2}(p) \\
 &= 2d\lambda_z [2 \log |z - p|] c_X^{-2}(p) \\
 &= 2[\{p\}] c_X^{-2}(p),
 \end{aligned} \tag{4.1}$$

where  $d\lambda_z$  is the Lebesgue measure with respect to  $z$ . Corollary 1.9 and equality (4.1) tell us that there exists a holomorphic section  $F$  of  $mK_X$  on  $X$ , such that  $F|_p = (dw)^m$  and

$$\int_X \sqrt{-1} |F|_{h^m}^2 \omega \leq \pi \left(1 + \frac{1}{\delta}\right) 2|(dw)^m|_{h^m}^2 c_X^{-2}(p).$$

Since

$$|\kappa_X(p, p)|_{h^m} \geq \frac{2|f|_{h^m}^2|_p}{\int_X |f|_{h^m}^2 \omega},$$

for any nonzero holomorphic section  $f$  of  $mK_X$  on  $X$ , we obtain the theorem.

#### 4.2 Proof of Theorem 1.11

As  $X$  is a complex torus, it is known that  $c_1(L) = b\omega$ ,  $b$  is a positive constant. It is known that

$$\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} g(\cdot, q)|_{X \setminus q} = -a\omega,$$

where  $a$  is a positive constant.

Note that

$$\int_X \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} g(\cdot, q) = 0 \quad \text{and} \quad \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} g(\cdot, q) = [\{q\}] - a\omega,$$

then we have

$$\int_X a\omega = 1.$$

Note that

$$\int_X b\omega = \int_X c_1(L) = d,$$

then it is clear that

$$\frac{2a}{b} = \frac{2}{d}.$$

Let  $\Psi = 2g(\cdot, p)$ . We choose

$$\delta = \frac{d}{2} - 1,$$

then it is clear that

$$\Psi \in \Delta_{h_L, \delta} \quad \text{and} \quad 1 + \frac{1}{\delta} = 1 + \frac{1}{\frac{d}{2} - 1}.$$

By the same method as in equality (4.1), we have

$$\omega[\Psi] = 2[\{p\}] c_X^{-2}(p). \tag{4.2}$$



Corollary 1.9 and equality (4.1) tell us that there exists a holomorphic section  $F$  of  $K_X \otimes L$  on  $X$ , such that  $F|_p = f(0)$  and

$$\int_X \sqrt{-1} |F|_{h_L}^2 \omega \leq \pi \left(1 + \frac{1}{\delta}\right) 2|f(0)|_{h_L}^2 c_X^{-2}(p).$$

Since

$$|\kappa_{X,d}(p, p)|_{h_L} \geq \frac{2|f|_{h_L}^2|_p}{\int_X |f|_{h_L}^2 \omega},$$

for any nonzero holomorphic section  $f$  of  $K_X \otimes L$  on  $X$ , we obtain the theorem.

### 4.3 Proof of Corollary 1.15

Note that  $2G_\Omega(\xi, z) - \log |\xi - z|^2$  is smooth with respect to  $\xi$ , on the coordinate neighborhood of  $z$ , when  $\Omega$  is a Riemann surface admitting the Green function  $G_\Omega(\xi, z)$ .

By Lemma 2.7, we have

$$d\lambda_z[2G_\Omega(\xi, z)] = d\lambda_z[\log |\xi - z|^2 + (2G_\Omega(\xi, z) - \log |\xi - z|^2)] = e^{-(2G_\Omega(\xi, z) - \log |\xi - z|^2)} \delta_z. \quad (4.3)$$

Let  $w$  be the local coordinate of a neighborhood of  $z_0$ . Then extended Suita conjecture becomes

$$(c_\beta(z_0))^2 |dw|^2 \leq \pi \rho(z_0) \kappa_{\Omega, \rho}(z_0).$$

Let  $M = \Omega$ ,  $\varphi = 2h$ ,  $S = z_0$ , and  $\Psi(z) = 2G_\Omega(z, z_0)$  in Theorem 1.12. Then the theorem and equality (4.3) tell us that there exists a holomorphic  $(1, 0)$ -form  $F$  on  $\Omega$ , such that  $F|_{z_0} = dw$  and

$$\int_\Omega \sqrt{-1} \rho F \wedge \bar{F} \leq \frac{2C\pi\rho(z_0)}{(c_\beta(z_0))^2}.$$

Since

$$\kappa_\Omega(z_0) \geq \frac{2f \otimes \bar{f}|_{z_0}}{\sqrt{-1} \int_\Omega \rho f \wedge \bar{f}}$$

for any nonzero holomorphic  $(1, 0)$ -form  $f$  on  $\Omega$ , we obtain Corollary 1.15.

### 4.4 Proof of Proposition 1.19

As  $C(\Omega, z_0) := \frac{c_\beta(z_0)^2 |dz|^2}{\pi \kappa_\Omega(z_0)}$ , by arguments in the proof of Corollary 1.15, it is clear that there exists a holomorphic  $(1, 0)$ -form  $F$  on  $\Omega$  satisfying

$$\int_\Omega \sqrt{-1} F \wedge \bar{F} = C(\Omega, z_0) \pi \int_{z_0} |F|^2 dV_\Omega[2G_\Omega(\cdot, z_0)], \quad (4.4)$$

and  $F|_{z_0} \neq 0$ .

By Theorem 2.1 in [15], there exists a holomorphic embedding  $f : \Omega \rightarrow \Delta$  such that  $f(z_0) = 0$  and  $\Delta(0, s_\Omega(z_0)) \subset f(\Omega)$ .

By the submean value inequality of plurisubharmonic function on  $\Delta(0, s_\Omega(z_0))$ , we have

$$\int_{f(\Omega)} \sqrt{-1} f_* F \wedge f_* \bar{F} \geq \pi s_\Omega^2(z_0) \int_{f(z_0)} |f_* F|^2 dV_{f(\Omega)}[\log |w|^2].$$

As  $\int_{f(\Omega)} \sqrt{-1} f_* F \wedge f_* \bar{F} = \int_\Omega \sqrt{-1} F \wedge \bar{F}$ , it is clear that

$$\int_\Omega \sqrt{-1} F \wedge \bar{F} \geq \pi s_\Omega^2(z_0) \int_{f(z_0)} |f_* F|^2 dV_{f(\Omega)}[\log |w|^2],$$

where  $w$  is the coordinate of  $\Delta \subset \mathbb{C}$ .

By equality (4.4), we obtain that

$$\pi C(\Omega, z_0) \int_{z_0} |F|^2 dV_\Omega[2G_\Omega(\cdot, z_0)] \geq \pi s_\Omega^2(z_0) \int_{z_0} |f_* F|^2 dV_{f(\Omega)}[\log |w|^2].$$

As  $\log |w|^2 - f_* 2G_\Omega(\cdot, z_0)$  is a negative smooth function on  $\Omega$ , by Lemma 2.7, we have

$$\int_{f(z_0)} |f_* F|^2 dV_{f(\Omega)}[f_* 2G_\Omega(\cdot, z_0)] \leq \int_{f(z_0)} |f_* F|^2 dV_{f(\Omega)}[\log |w|^2].$$

Note that

$$\int_{z_0} |F|^2 dV_\Omega[2G_\Omega(\cdot, z_0)] = \int_{f(z_0)} |f_* F|^2 dV_{f(\Omega)}[f_* 2G_\Omega(\cdot, z_0)],$$

and  $1 \geq C(\Omega, z_0)$ , then we have  $C(\Omega, z_0) \geq s_\Omega^2(z_0)$ .

When  $\lim_{z \rightarrow \partial\Omega} s_\Omega(z) = 1$ , it is clear that  $\lim_{z \rightarrow \partial\Omega} C(\Omega, z) = 1$ .

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